Stochastic target problems and pricing under risk constraints

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Joint works with R. Elie, M. N. Dang, L. Moreau, M. Nutz, N. Touzi and T. N. Vu
Motivation
General setting

- $\phi$: trading strategy
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- $Y^\phi_y$: wealth process, valued in $\mathbb{R}$, initial wealth $y$
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- $Y^\phi_y$: wealth process, valued in $\mathbb{R}$, initial wealth $y$
- $X^\phi$: stocks, factors, valued in $\mathbb{R}^d$

Target: $\mathbb{E}[G(X^\phi(T), Y^\phi_y(T))] \geq p$, $p \in \mathbb{R}$, $G: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$

Constraint: $(X^\phi, Y^\phi_y) \in O$ up to $T$ ($O: t \mapsto O(t) \subset \mathbb{R}^{d+1}$)

Price under risk constraint: $\inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi_y) \in O \text{ and } \mathbb{E}[G(X^\phi(T), Y^\phi_y(T))] \geq p \right\}$. 
General setting

- $\phi$: trading strategy

- $Y^\phi_y$: wealth process, valued in $\mathbb{R}$, initial wealth $y$

- $X^\phi$: stocks, factors, valued in $\mathbb{R}^d$

- Target: $\mathbb{E}\left[G(X^\phi(T), Y^\phi_y(T))\right] \geq p$, $p \in \mathbb{R}$, $G: \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$
General setting

□ $\phi$ : trading strategy

□ $Y^\phi_y$ : wealth process, valued in $\mathbb{R}$, initial wealth $y$

□ $X^\phi$ : stocks, factors, valued in $\mathbb{R}^d$

□ Target : $\mathbb{E}\left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p$, $p \in \mathbb{R}$, $G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$

□ Constraint : $(X^\phi, Y^\phi_y) \in \mathcal{O}$ up to $T$ ($\mathcal{O} : t \mapsto \mathcal{O}(t) \subset \mathbb{R}^{d+1}$)
General setting

- $\phi$: trading strategy

- $Y_y^\phi$: wealth process, valued in $\mathbb{R}$, initial wealth $y$

- $X^\phi$: stocks, factors, valued in $\mathbb{R}^d$

- Target: $\mathbb{E} \left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p$, $p \in \mathbb{R}$, $G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$

- Constraint: $(X^\phi, Y_y^\phi) \in \mathcal{O}$ up to $T$ ($\mathcal{O} : t \mapsto \mathcal{O}(t) \subset \mathbb{R}^{d+1}$)

- Price under risk constraint:

$$\inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p \right\} .$$
Examples of dynamics: “usual” large investor model

- Control $\phi$: predictable process with values in $U \subset \mathbb{R}^d$.

\[
\begin{align*}
    dX^\phi &= \mu_X(X^\phi, \phi)dr + \sigma_X(X^\phi, \phi)dW \\
    dY^\phi &= \phi' \mu_X(X^\phi, \phi)dr + \phi' \sigma_X(X^\phi, \phi)dW.
\end{align*}
\]

- $X^\phi = \text{stocks}$, $Y^\phi = \text{wealth}$, $\phi = \text{number of stocks in the portfolio}$. 
Examples of dynamics: proportional transaction costs

- Control $\phi$ adapted non-decreasing process (component by component)

\[
X^1(s) = x^1 + \int_t^s X^1(r) \mu dr + \int_t^s X^1(r) \sigma dW^1_r
\]

\[
X^{2,\phi}(s) = x^2 + \int_t^s \frac{X^{2,\phi}(r)}{X^1(r)} dX^1(r) - \int_t^s d\phi^1_r + \int_t^s d\phi^2_r
\]

\[
Y^\phi(s) = y + \int_t^s (1 - \lambda) d\phi^1_r - \int_t^s (1 + \lambda) d\phi^2_r.
\]

- $X^1$ = stock, $X^{2,\phi}$ = value invested in the stock, $Y^\phi$ = value invested in cash
- $\phi^1_t$ = cumulated amount of stocks sold, $\phi^2_t$ = cumulated amount of stocks bought.
- $\lambda > 0$ : proportional transaction cost coefficient.
Examples of dynamics: model with immediate proportional price impact

- Control $\phi$ adapted non-decreasing process (component by component)

$$dX^\phi = \mu_X(X^\phi)dr + \sigma_X(X^\phi)dW + \beta_X(X^\phi)d\phi$$

$$dY^\phi = -X^\phi d\phi .$$

- $X^\phi = \text{stock}$, $Y^\phi = \text{wealth}$, $d\phi = \text{number of stocks bought at time } t$.
- $\beta_X = \text{immediate impact factor}$.

Examples of dynamics: model with immediate non-proportional price impact

Control $\phi = \sum_{i \geq 1} \xi_i 1_{[\tau_i, \tau_{i+1})}$ adapted

\[
dX^{1,\phi} = \mu_X(X^{\phi})dr + \sigma_X(X^{\phi})dW + \sum_{i \geq 1} \beta_X(X^{\phi}, \Delta \phi) 1_{\tau_i}
\]

\[
dX^{2,\phi} = \sum_{i \geq 1} \Delta \phi 1_{\tau_i}
\]

\[
dY^\phi = -\sum_{i \geq 1} \beta_Y(X^{\phi}, \Delta \phi) 1_{\tau_i}.
\]

$\Rightarrow X^{1,\phi} =$ stock, $X^{2,\phi} =$ number of stocks in the portfolio, $Y^\phi =$ cash account, $\Delta \phi_{\tau_i} =$ number of stocks bought/sold at time $\tau_i$.

$\beta_X =$ immediate impact factor, $\beta_Y =$ buying/selling cost.
Other possible dynamics

- Dynamics with jumps (finance/insurance) : L. Moreau, B.
Other possible dynamics

- Dynamics with jumps (finance/insurance) : L. Moreau, B.
- Any mixed control type problems.
Examples of constraints: super-hedging

- Problem:
  \[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p \right\}. \]

- Take
  \[ \mathcal{O} := \mathbb{R}^{d+1}1_{[0,T)} + 1_{\{T\}} \{(x, y) : y \geq g(x)\}, \ G = 0 \text{ and } p = 0. \]

- Super-hedging of an European option:
  \[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y^\phi_y(T) \geq g(X^\phi(T)) \right\}. \]
Examples of constraints: super-hedging

- Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p \right\} . \]

- Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \ G(x, y) = 1_{y \geq g(x)} \text{ and } p = 1. \]

- Super-hedging of an European option:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y_y^\phi(T) \geq g(X^\phi(T)) \right\} . \]
Examples of constraints: super-hedging of American option

- Problem:
  
  \[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi_y) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p \right\}. \]

  \[ \mathcal{O} := \{ (x, y) : y \geq g(x) \}, \ G = 0 \text{ and } p = 0. \]

- Super-hedging of an American option:
  
  \[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y^\phi_y \geq g(X^\phi) \text{ up to } T \right\}. \]
Examples of constraints: P&L-hedging

Problem:

\[ \nu := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y^\phi(T)) \right] \geq p \right\}. \]

Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \ G^i(x, y) = 1_{y - g(x) \geq -\gamma^i} \text{ and } p^i \in (0, 1). \]

with

\[ \mathbb{P} \left[ Y^\phi(T) - g(X^\phi(T)) \geq -\gamma^i \right] \geq p^i \text{ with } \gamma^i \uparrow, \ p^i \uparrow \]

\[ \Rightarrow \text{ P&L constraint (see B. and Vu 2011)}. \]
Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X_\phi, Y_\phi) \in O \text{ and } \mathbb{E} \left[ G(X_\phi(T), Y_\phi(T)) \right] \geq p \right\}. \]

Take

\[ O := \mathbb{R}^{d+1}, \quad G(x, y) = -\ell([y - g(x)]^-) \text{ and } p < 0. \]

⇒ Shortfall-hedging of European option.
Examples of constraints: indifference pricing

Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p \right\}. \]

Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \quad G(x, y) = U(y_0 + y - g(x)) \text{ and } p := \sup_{\phi} \mathbb{E} \left[ U(Y^\phi_{t,x,y_0}(T)) \right]. \]

\[ \Rightarrow \text{Utility indifference price}. \]
Aim
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- Provide a PDE characterization in the (Markovian) situations where

  - markets are incomplete
  - markets have frictions
  - models without any notion of martingale measure
    
    Ex: guaranteed VWAP liquidation contracts.

- Based on a "risk" criteria.

- We want a direct approach:
  
  - one (non-linear) pricing equation
  
  - no-numerical inversion procedure

  $\inf_{\phi} \max_{\phi} E \left[ G(\mathbf{X}, \mathbf{Y}, \phi) \right] \geq \phi = v$.

- If one can allow for high dimensions: include liquid options as assets ⇒ automatically calibrated.
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- We want a direct approach:
  - one (non-linear) pricing equation
  - no-numerical inversion procedure
    \[
    \left( \inf_Y \max_{\phi} \mathbb{E} \left[ G(X^{\phi}, Y^{\phi}_Y) \right] \right) \geq p = v.
    \]
Aim

- Provide a PDE characterization in the (Markovian) situations where
  - markets are incomplete
  - markets have frictions
  - models without any notion of *martingale measure*. Ex: guaranteed VWAP liquidation contracts.
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- We want a direct approach:
  - one (non-linear) pricing equation
  - no-numerical inversion procedure
    \[
    \inf_y \max_\phi \mathbb{E} \left[ G(X_\phi, Y_\phi^y) \right] \geq p = v.
    \]
- If one can allow for high dimensions: include liquid options as assets \(\Rightarrow\) automatically calibrated.
Geometric Dynamic Programming

- Problem extension: \( Z_{t,z}^{\phi} = (X_{t,x}^{\phi}, Y_{t,x,y}^{\phi}) \)
Geometric Dynamic Programming

- Problem extension: \( Z_{t,x}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi) \)

\[
v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\}.
\]
Problem extension: \( Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi) \)

\[ v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in O \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}(T)) \right] \geq p \right\} . \]

Assumption: \( y' \geq y \) and \((x, y) \in O \Rightarrow (x, y') \in O, t \mapsto O(t) \) is right-continuous and \( G \uparrow \) in \( y \).
The $\mathbb{P} - \text{a.s.}$ case

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$$v(t, x) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] \right\}.$$
The $\mathbb{P} - \text{a.s.}$ case

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

\[ v(t, x) : = \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] \right\}. \]

- Theorem: For all $\phi$ and $\theta \in \mathcal{T}_{[t,T]}$:

GDP1:

\[ Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \]

GDP2:

\[ y < v(t, x) \Rightarrow \mathbb{P} \left[ Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \right. \text{ and } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, \theta] \left. \right] < 1 \]
The $\mathbb{P} - \text{a.s.}$ case

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

\[ v(t, x) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] \right\}. \]

- **Theorem**: For all $\phi$ and $\theta \in \mathcal{T}_{[t,T]}$:

  GDP1:

  \[ Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \implies Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \]

  GDP2:

  \[ y < v(t, x) \implies \mathbb{P} \left[ Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, \theta] \right] < 1 \]

- First introduced by Soner and Touzi for super-hedging under Gamma constraints. Extended to American type constraints: obstacle version of B. and Vu.
Constraints in expectations

Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] , \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\}$. 
Constraints in expectations

- Problem extension: \( Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi) \)

\[
\nu(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\}.
\]

- Theorem: For all \( \phi \) and \( \theta \in \mathcal{T}_{[t, T]} \):

GDP1:

\[
Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq \nu(\theta, X_{t,x}^\phi(\theta), p).
\]
Constraints in expectations

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- Theorem: For all $\phi$ and $\theta \in \mathcal{T}_{[t,T]}$:

GDP1:

$$Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta), P_{t,p}(\theta))$$

with

$$P_{t,p}(\theta) := \mathbb{E}\left[ G(Z_{t,z}^\phi(T)) \mid \mathcal{F}_\theta \right]$$
Constraints in expectations

Problem extension: \( Z_{t,z}^{\phi} = (X_{t,x}^{\phi}, Y_{t,x,y}^{\phi}) \)

\[
\nu(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^{\phi} \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^{\phi}(T)) \right] \geq p \right\}.
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Theorem: For all \( \phi \) and \( \theta \in \mathcal{T}_{[t, T]} : \)

\( GDP1 : \)

\[
Z_{t,z}^{\phi} \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^{\phi}(\theta) \geq \nu(\theta, X_{t,x}^{\phi}(\theta), P_{t,p}(\theta))
\]

with

\[
P_{t,p}(\theta) := \mathbb{E} \left[ G(Z_{t,z}^{\phi}(T)) \mid \mathcal{F}_{\theta} \right] = p + \int_{t}^{\theta} \alpha_s dW_s,
\]

if \( \mathcal{F}_t = \sigma(W_s, s \leq t) \).
Constraints in expectations

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$$v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\} .$$
**Constraints in expectations**

- **Problem extension:** \( Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi) \)

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\]

- **Problem reduction:** (B., Elie and Touzi) For all \( \phi \):

\[
Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \text{ and } \mathbb{E}\left[ G(Z_{t,z}^\phi(T)) \right] \geq p
\]

if and only if \( \exists \alpha \) such that

\[
(Z_{t,z}^\phi, P_{t,p}^\alpha) \in \mathcal{O} \times \mathbb{R} \text{ on } [t, T] \text{ and } G(Z_{t,z}^\phi(T)) \geq P_{t,p}^\alpha(T)
\]

with

\[
P_{t,p} := \mathbb{E}\left[ G(Z_{t,z}^\phi(T)) \mid \mathcal{F} \right] = p + \int_t^T \alpha_s dW_s.
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Constraints in expectations

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□ Problem reduction: (B., Elie and Touzi) For all \( \phi : \)

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if and only if \( \exists \alpha \) such that

\[ (Z_{t,z}^{\phi}, P_{t,p}^{\alpha}) \in \mathcal{O} \times \mathbb{R} \text{ on } [t, T] \text{ and } G(Z_{t,z}^{\phi}(T)) \geq P_{t,p}^{\alpha}(T) \]

with

\[ P_{t,p} := \mathbb{E}\left[ G(Z_{t,z}^{\phi}(T)) \mid \mathcal{F} \right] = p + \int_t^T \alpha_s dW_s . \]

□ Can use the GDP with an increased controlled process.
PDE derivation

Previous works

• Soner and Touzi: Brownian filtration and bounded controls (apart from particular cases in finance).
  \( P - a.s. \) criteria.

• B.: Jump diffusion with bounded control and locally bounded jumps.
  \( P - a.s. \) criteria.

• B., Elie and Touzi: Brownian filtration with unbounded controls. Criteria in expectation (concentrating on the case of a criteria in expectation).

• B. and Vu: "American" case.

• Moreau: Extension of B., Elie and Touzi to jump diffusions.

• B. and Dang: controls of bounded variations types.
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The general model

- Set of controls: $\nu \in \mathcal{U}$, predictable, square integrable, with values in $U \subset \mathbb{R}^d$. 
The general model

- Set of controls: $\nu \in \mathcal{U}$, predictable, square integrable, with values in $\mathcal{U} \subset \mathbb{R}^d$.

- Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

  \[
  \begin{align*}
  dX^\nu &= \mu_X(X^\nu, \nu)dr + \sigma_X(X^\nu, \nu)dW \\
  dY^\nu &= \mu_Y(Z^\nu, \nu)dr + \sigma_Y(Z^\nu, \nu)dW .
  \end{align*}
  \]
The general model

- Set of controls: $\nu \in \mathcal{U}$, predictable, square integrable, with values in $\mathcal{U} \subset \mathbb{R}^d$.

- Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:
  
  \begin{align*}
  dX^\nu &= \mu_X(X^\nu, \nu)dr + \sigma_X(X^\nu, \nu)dW \\
  dY^\nu &= \mu_Y(Z^\nu, \nu)dr + \sigma_Y(Z^\nu, \nu)dW .
  \end{align*}

- Problem:
  
  $\nu(t, x, p) := \inf \{ y : \exists \nu \in \mathcal{U} / Z^\nu_{t,x,y} \in \mathcal{O} , \ \mathbb{E} [ G(Z^\nu_{t,x,y}(T)) ] \geq p \}$
The general model

- Set of controls: \( \nu \in \mathcal{U} \), predictable, square integrable, with values in \( \mathcal{U} \subset \mathbb{R}^d \).

- Dynamics of \( Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R} \):

\[
\begin{align*}
  dX^\nu &= \mu_X(X^\nu, \nu)\,dr + \sigma_X(X^\nu, \nu)dW \\
  dY^\nu &= \mu_Y(Z^\nu, \nu)\,dr + \sigma_Y(Z^\nu, \nu)dW.
\end{align*}
\]

- Problem:

\[
\nu(t, x, p) := \inf \left\{ y : \exists \nu \in \mathcal{U} / \, Z^\nu_{t,x,y} \in \mathcal{O}, \, \mathbb{E} \left[ G(Z^\nu_{t,x,y}(T)) \right] \geq p \right\}
\]

- Reduction: A set of predictable square integrable processes

\[
\inf \left\{ y : \exists (\nu, \alpha) \in \mathcal{U} \times \mathcal{A} / \, Z^\nu_{t,x,y} \in \mathcal{O}, \, G(Z^\nu_{t,x,y}(T)) \geq P^\alpha_{t,p}(T) \right\}.
\]
Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists (\nu, \alpha)$ such that $Z_{t,z}^\nu \in \mathcal{O}$ on $[t, T]$ and $G(Z_{t,x,y}^\nu(T)) \geq P_{t,p}^\alpha(T)$. 

Formal derivation of the PDE
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x, p) \), \( \exists (\nu, \alpha) \) such that \( Z_{t,z}^{\nu} \in \mathcal{O} \) on \([t, T]\) and \( G(Z_{t,x,y}^{\nu}(T)) \geq P_{t,p}^{\alpha}(T) \).

Then \( Y_{t,z}^{\nu}(t+) \geq v(t+, X_{t,x}^{\nu}(t+), P_{t,p}^{\alpha}(t+)) \) and
Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists (\nu, \alpha)$ such that $Z_{t,z}^{\nu} \in \mathcal{O}$ on $[t, T]$ and $G(Z_{t,x,y}^{\nu}(T)) \geq P_{t,p}^{\alpha}(T)$.

Then $Y_{t,z}^{\nu}(t+) \geq v(t+, X_{t,x}^{\nu}(t+), P_{t,p}^{\alpha}(t+))$ and

$$\left( \mu_Y(z, \nu_t) - \mathcal{L}_{X, P}^{\nu_t, \alpha_t} v(t, x, p) \right) dt$$

$$\geq (\sigma_Y(z, \nu_t) - D_x v(t, x, p) \sigma_X(x, \nu_t) - D_p v(t, x, p) \alpha_t) dW_t$$
Formal derivation of the PDE

Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists (\nu, \alpha)$ such that $Z^\nu_{t, z} \in \mathcal{O}$ on $[t, T]$ and $G(Z^\nu_{t, x, y}(T)) \geq P^\alpha_{t, p}(T)$.

Then $Y^\nu_{t, z}(t+) \geq v(t+, X^\nu_{t, x}(t+), P^\alpha_{t, p}(t+))$ and

$$
\left( \mu_Y(z, \nu_t) - L^\nu_{X, P} v(t, x, p) \right) dt \\
\geq (\sigma_Y(z, \nu_t) - D_x v(t, x, p) \sigma_X(x, \nu_t) - D_p v(t, x, p) \alpha_t) dW_t
$$

This implies that:

$$
\sigma_Y(x, v(t, x, p), \nu_t) = D_x v(t, x, p) \sigma_X(x, \nu_t) + D_p v(t, x, p) \alpha_t
$$
Formal derivation of the PDE

Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists (\nu, \alpha)$ such that $Z_{t,z}^\nu \in \mathcal{O}$ on $[t, T]$ and $G(Z_{t,x,y}^\nu(T)) \geq P_{t,p}^\alpha(T)$.

Then $Y_{t,z}^\nu(t+) \geq v(t+, X_{t,x}^\nu(t+), P_{t,p}^\alpha(t+))$ and

$$
\left( \mu_Y(z, \nu_t) - \mathcal{L}_{X,P}^{\nu_t,\alpha_t} v(t, x, p) \right) dt \\
\geq (\sigma_Y(z, \nu_t) - D_x v(t, x, p) \sigma_X(x, \nu_t) - D_p v(t, x, p) \alpha_t) dW_t
$$

This implies that :

$\sigma_Y(x, v(t, x, p), \nu_t) = D_x v(t, x, p) \sigma_X(x, \nu_t) + D_p v(t, x, p) \alpha_t$

And that :

$\mu_Y(x, v(t, x, p), \nu_t) - \mathcal{L}_{X,P}^{\nu_t,\alpha_t} v(t, x, p) \geq 0.$
Formal derivation of the PDE

Set

\[ Fv := \sup \{ \mu_Y(\cdot, v) - \mathcal{L}^a X, P v, (u, a) \in Nv \} \]

with

\[ Nv := \{(u, a) \in U \times \mathbb{R}^d : \sigma_Y(\cdot, v, u) = D_x v \sigma_X(\cdot, u) + D_p v a \} \].

PDE characterization in the interior of the domain

\[ Fv = 0 \text{ on } (t, x, v(t, x)) \in \text{int}(D) \]

where \( D := \{(t, x, y) : (x, y) \in \mathcal{O}(t) \} \).
**PDE on the space boundary** $(x, y) \in \partial \mathcal{O}(t)$

Domain is

$$D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.$$
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).
**PDE on the space boundary** \((x, y) \in \partial \mathcal{O}(t)\)

Domain is 

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

**Assumption**: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z_{t,z}^\nu(t)) \geq 0\) if \((t, z) \in \partial D\).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is
\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

**Assumption** : \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in int\((D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z_{t,z}(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies :
\[
\mathcal{L}^{\nu_t}_Z \delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y, \nu_t) = 0
\]
when \((t, z) \in \partial D\).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

This leads to the definition of

\[
N^{\text{in}}_v : = \{(u, a) \in Nv : D\delta(\cdot, v)\sigma_Z(\cdot, v, u) = 0\}
\]

\[
F^{\text{in}}_v : = \sup_{(u, a) \in N^{\text{in}}_v} \min \left\{ \mu_Y(\cdot, v, u) - \mathcal{L}^{u,a}_{X,P} v, \mathcal{L}^u_Z \delta(\cdot, v) \right\}
\]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

This leads to the definition of

\[
N^{\text{in}}_v := \{ (u, a) \in N_v : D \delta(\cdot, v) \sigma_Z(\cdot, v, u) = 0 \}
\]

\[
F^{\text{in}}_v := \sup_{(u, a) \in N^{\text{in}}_v} \min \left\{ \mu_Y(\cdot, v, u) - \mathcal{L}^{u,a}_{X,P} v, \mathcal{L}^u_Z \delta(\cdot, v) \right\}
\]

Then, the PDE on the boundary reads

\[
F^{\text{in}}_0 v = 0 \quad \text{on} \quad (t, x, v(t, x)) \in \partial D.
\]
PDE derivation for models with controls of bounded variation type
The general model

- Set of controls: \( L \in \mathcal{L} \) set of continuous non-decreasing \( \mathbb{R}^d \)-valued adapted processes \( L \) s.t. \( \mathbb{E} \left[ |L|^2_T \right] < \infty \).
The general model

- Set of controls: \( L \in \mathcal{L} \) set of continuous non-decreasing \( \mathbb{R}^d \)-valued adapted processes \( L \) s.t. \( \mathbb{E} \left[ |L|_{T}^{2} \right] < \infty \).

- Dynamics of \( Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R} \):
  \[
  dX^L = \mu_X(X^L)dr + \sigma_X(X^L)dW + \beta_X(X^L)dL \\
  dY^L = \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.
  \]
The general model

☐ Set of controls: $L \in \mathcal{L}$ set of continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E}\left[|L|_T^2\right] < \infty$.

☐ Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

$$dX^L = \mu_X(X^L)dr + \sigma_X(X^L)dW + \beta_X(X^L)dL$$
$$dY^L = \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.$$  

☐ Problem:

$$v(t, x, p) := \inf \{ y : \exists L \in \mathcal{L} / Z^L_{t,x,y} \in \mathcal{O} , \mathbb{E}[G(Z^L_{t,x,y}(T))] \geq p \}$$
The general model

- **Set of controls:** $L \in \mathcal{L}$ set of continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} \left[ |L|^2_T \right] < \infty$.

- **Dynamics of** $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}:
  \begin{align*}
  dX^L &= \mu_X(X^L)dr + \sigma_X(X^L)dW + \beta_X(X^L)dL \\
  dY^L &= \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.
  \end{align*}

- **Problem:**
  \[ v(t, x, p) := \inf \{ y : \exists L \in \mathcal{L} / Z^L_{t,x,y} \in \mathcal{O} \text{, } \mathbb{E} \left[ G(Z^L_{t,x,y}(T)) \right] \geq p \} \]

- **Reduction:** $A$ set of predictable square integrable processes
  \[ \inf \{ y : \exists (L, \alpha) \in \mathcal{L} \times A / Z^L_{t,x,y} \in \mathcal{O} \text{, } G(Z^L_{t,x,y}(T)) \geq P^\alpha_{t,p}(T) \} . \]
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x, p) \), \( \exists (L, \alpha) \) such that \( Z^L_{t,z} \in \mathcal{O} \) on \([t, T]\) and \( G(Z^L_{t,x,y}(T)) \geq P^\alpha_{t,p}(T) \).
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x, p) \), \( \exists \ (L, \alpha) \) such that \( Z_{t,z}^L \in \mathcal{O} \) on \( [t, T] \) and \( G(Z_{t,x,y}^L(T)) \geq P_{t,p}^\alpha(T) \).

Then \( Y_{t,z}^L(t+) \geq v(t+, X_{t,x}^L(t+), P_{t,p}^\alpha(t+)) \) and
Assume that \( \nu \) is smooth and the inf is achieved.

For \( y = \nu(t, x, p) \), \( \exists (L, \alpha) \) such that \( Z^L_{t, z} \in \mathcal{O} \) on \([t, T]\) and \( G(Z^L_{t, x, y}(T)) \geq P^\alpha_{t, p}(T) \).

Then \( Y^L_{t, z}(t+) \geq \nu(t+, X^L_{t, x}(t+), P^\alpha_{t, p}(t+)) \) and

\[
(\mu_Y(z) - \mathcal{L}^\alpha_{X, P} \nu(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x \nu(t, x, p) \sigma_X(x) - D_p \nu(t, x, p) \alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x \nu(t, x, p) \beta_X(x)) \, dL_t
\]
Formal derivation of the PDE

\[
(\mu_Y(z) - \mathcal{L}_{X,P}^\alpha v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p)\beta_X(x)) \, dL_t
\]
Formal derivation of the PDE

\[
\begin{align*}
(\mu_Y(z) - \mathcal{L}^\alpha_{\mathcal{X}, p} v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p)\beta_X(x)) \, dL_t
\end{align*}
\]

Ok if  \( \mu_Y(x, v(t, x, p)) - \mathcal{L}^\alpha_{\mathcal{X}, p} v(t, x, p) \geq 0 \)

with  \( \sigma_Y(x, v(t, x, p)) = D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha. \)
Formal derivation of the PDE

\[
(\mu_Y(z) - \mathcal{L}_{X,P}^\alpha v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p)\beta_X(x)) \, dL_t
\]

Ok if \( \mu_Y(x, v(t, x, p)) - \mathcal{L}_{X,P}^\alpha v(t, x, p) \geq 0 \)

with \( \sigma_Y(x, v(t, x, p)) = D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha. \)

Or \( (\beta_Y(x, v(t, x, p)) - D_x v(t, x, p)\beta_X(x)) \ell > 0 \)

with \( \ell \in \Delta_+ := \partial B_1(0) \cap \mathbb{R}_+^d. \)
Formal derivation of the PDE

Set

\[ F_v := \sup \left\{ \mu_Y(\cdot, v) - \mathcal{L}_{X,P}^\alpha v, \: \alpha \in N_v \right\} \]
\[ G_v := \max \left\{ [\beta_Y(\cdot, v) - D_x v(t, x)\beta_X(x)] \ell, \: \ell \in \Delta_+ \right\} \]

with

\[ N_v := \{ \alpha : \sigma_Y(\cdot, v) = D_x v \sigma_X + D_P v \alpha \} \]
\[ \Delta_+ := \mathbb{R}_+^d \cap \partial B_1(0). \]

PDE characterization in the interior of the domain

\[ \max \{ F_v, \: G_v \} = 0 \text{ on } (t, x, v(t, x)) \in \text{int}(D) \]

where \( D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\} \).
PDE on the space boundary $(x, y) \in \partial \mathcal{O}(t)$

Domain is

$$D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.$$
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

**Assumption:** \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z^L_{t,z}(t)) \geq 0\) if \((t, z) \in \partial D\).
PDE on the space boundary \((x, y) \in \partial O(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in O(t)\}. \]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z_{t,z}^L(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies : either

\[ L_Z \delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y) = 0 \]
**PDE on the space boundary** \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

**Assumption:** \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z_{t,z}^L(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies: or

\[
\max\{D\delta(t, x, y)\beta_z(x, y)\ell, \ \ell \in \Delta_+\} > 0.
\]
PDE on the space boundary $(x, y) \in \partial \mathcal{O}(t)$

The GDP and the need for a reflexion on the boundary leads to the definition of

- $N^{in} \nu := \{ \alpha \in N\nu : D\delta(\cdot, \nu)\sigma_Z(\cdot, \nu) = 0 \}$
- $F^{in} \nu := \sup \min_{\alpha \in N^{in} \nu} \{ \mu_Y(\cdot, \nu) - \mathcal{L}_{X, P \nu}^\alpha, \mathcal{L}_Z \delta(\cdot, \nu) \}$
- $G^{in} \nu := \max \min_{\ell \in \Delta_+} \{ [\beta_Y(\cdot, \nu) - D_x \nu \beta_X] \ell, D\delta(\cdot, \nu)\beta_Z(\cdot, \nu) \ell \}$
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

The GDP and the need for a reflexion on the boundary leads to the definition of

\[
\begin{align*}
N^{in} v & := \{ \alpha \in \mathcal{N}v : D \delta(\cdot, v) \sigma Z(\cdot, v) = 0 \} \\
F^{in} v & := \sup_{\alpha \in \mathcal{N}^{in} v} \min \{ \mu_Y(\cdot, v) - \mathcal{L}_{X,P}^\alpha v, \mathcal{L}_Z \delta(\cdot, v) \} \\
G^{in} v & := \max_{\ell \in \Delta^+} \min \{ [\beta_Y(\cdot, v) - D_X v \beta_X]\ell, D \delta(\cdot, v) \beta_Z(\cdot, v)\ell \}
\end{align*}
\]

Then, the PDE on the boundary reads

\[
\max\{ F^{in}_0 v, G^{in} v \} = 0 \text{ on } (t, x, v(t, x)) \in \partial D.
\]
Example

Pricing of the VWAP-guaranteed liquidation contract
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
- Ensure that will guarantee a mean selling price of $\gamma \times$ the mean selling price of the market.
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
- Ensure that will guarantee a mean selling price of $\gamma \times$ the mean selling price of the market.
- What is the price of the guarantee?
The VWAP guaranted pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \# \text{ of sold stocks.}$
- Price dynamics:

\[ dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t \]
The VWAP guaranteed pricing problem

- Controls: $L$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:

\[ dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t \]

- Cumulated gain from liquidation: $dY^L = X^{L,1} dL_t$
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:

$$dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t$$

- Cumulated gain from liquidation: $dY^L = X^{L,1} dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1} d\vartheta$. 
The VWAP guaranted pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:

$$dX^{L,1} = X^{L,1} \mu(X^{L,1}) dt + X^{L,1} \sigma(X^{L,1}) dW_t - X^{L,1} \beta(X^{L,1}(t)) dL_t$$

- Cumulated gain from liquidation: $dY^L = X^{L,1} dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1} d\vartheta$.
- Cumulated # of sold stocks: $X^{L,3} := L \in [\underline{\Lambda}, \bar{\Lambda}] \rightarrow \{K\}$
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:

$$dX^{L,1} = X^{L,1}\mu(X^{L,1})dt + X^{L,1}\sigma(X^{L,1})dW_t - X^{L,1}\beta(X^{L,1}(t))dL_t$$

- Cumulated gain from liquidation: $dY^L = X^{L,1}dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1}d\vartheta$.
- Cumulated # of sold stocks: $X^{L,3} := L \in [\underline{\Lambda}, \bar{\Lambda}] \to \{K\}$

- Risk constraint (with $\gamma \in (0, 1)$)

$$X^{L,3}_{t,x} \in [\underline{\Lambda}, \bar{\Lambda}] \text{ and } \mathbb{E} \left[ \ell \left( Y^{L}_{t,x,y}(T) - K\gamma X^{L,2}_{t,x}(T) \right) \right] \geq p \}.$$
The VWAP guaranted pricing problem

- Controls: \( L \uparrow \) adapted and continuous. \( L_t = \# \) of sold stocks.
- Price dynamics:

\[
dX^{L,1} = X^{L,1}_\mu(X^{L,1})dt + X^{L,1}_\sigma(X^{L,1})dW_t - X^{L,1}_\beta(X^{L,1}(t))dL_t
\]

- Cumulated gain from liquidation: \( dY^L = X^{L,1}_\xi dL_t \)
- Volume weighted market price: \( dX^{L,2} = X^{L,1}_d\vartheta \).
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- Pricing function (with \( \Psi(x, y) = \ell(y - \gamma Kx^2), \gamma > 0 \))

\[
v(t, x, p) := \inf\{y \geq 0 : \exists L \text{ s.t. } X^{L,3}_{t,x} \in [\Lambda, \bar{\Lambda}] , \, \mathbb{E} \left[ \Psi(Z^L_{t,x,y}(T)) \right] \geq p \} .
\]
**PDE characterization**

**Proposition** Under “good assumptions”, $v_\ast$ is a viscosity supersolution on $[0, T)$ of

$$\max \left\{ F\varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \text{ if } \Lambda \leq x^3 \leq \Lambda$$

and $v_\ast$ is a subsolution on $[0, T)$ of

$$\min \left\{ \varphi, \max \left\{ F\varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} \right\} = 0 \text{ if } \Lambda < x^3 < \Lambda$$

$$\min \left\{ \varphi, x^1 + \beta D_{x^1} \varphi - D_{x^3} \varphi \right\} = 0 \text{ if } \Lambda = x^3$$

$$\min \left\{ \varphi, F\varphi \right\} = 0 \text{ if } x^3 = \Lambda,$$

where

$$F\varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).$$

Moreover, $v_\ast(T, x, p) = v_\ast(T, x, p) = \Psi^{-1}(x, p)$. 

PDE characterization

Proposition Under “good assumptions”, \( v_* \) is a viscosity supersolution on \([0, T)\) of

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\]

Moreover, \( v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p) \).
The “good assumptions”

□ On \( \Lambda, \bar{\Lambda} \):

\[
\Lambda, \bar{\Lambda} \in C^1, \ \Lambda < \bar{\Lambda} \text{ on } [0, T), \ D\Lambda, D\bar{\Lambda} \in (0, M]
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The “good assumptions”

□ On $\Lambda, \bar{\Lambda}$:

$$\Lambda, \bar{\Lambda} \in C^1, \, \Lambda < \bar{\Lambda} \text{ on } [0, T), \, D\Lambda, D\bar{\Lambda} \in (0, M]$$

□ On the loss function $\ell$:

$$\exists \, \epsilon > 0 \text{ s.t. } \epsilon \leq D^-\ell, \, D^+\ell \leq \epsilon^{-1},$$

and

$$\lim_{r \to \infty} D^+\ell(r) = \lim_{r \to \infty} D^-\ell(r).$$
Control on the gradients

□ Proposition $v_*$ is a viscosity supersolution of

$$\min \{ D_p \varphi - \epsilon , (D_{x^1} \varphi - CD_p \varphi) 1_{x^1 > 0} , -D_{x^1} \varphi + CD_p \varphi \} = 0 \ (\ast)$$

and $v^*$ is a viscosity subsolution of

$$\max \{ -D_p \varphi + \epsilon , (D_{x^1} \varphi - CD_p \varphi) 1_{x^1 > 0} , -D_{x^1} \varphi + CD_p \varphi \} = 0 \ . \ (\ast\ast)$$

where $C$ is continuous and depends only on $x$. 
Control on the gradients

\[\Box \text{ Proposition } v_* \text{ is a viscosity supersolution of} \]

\[\min \left\{ D_p \varphi - \epsilon , (D_{x^1} \varphi - CD_p \varphi)1_{x^1 > 0} , -D_{x^1} \varphi + CD_p \varphi \right\} = 0 \quad (*)\]

and \(v^*\) is a viscosity subsolution of

\[\max \left\{ -D_p \varphi + \epsilon , (D_{x^1} \varphi - CD_p \varphi)1_{x^1 > 0} , -D_{x^1} \varphi + CD_p \varphi \right\} = 0 \quad (**)\]

where \(C\) is continuous and depends only on \(x\).

\[\Box \text{ Provides a control on the ratio } D_{x^1} \varphi / D_p \varphi \text{ in} \]

\[F \varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi)D_{(x^1, \varphi)}^2 \varphi \right) .\]
More controls on $v$

It also implies that $\exists \eta > 0$ s.t.

$0 \leq v(t, x, p) \leq \epsilon - 1 |p - \ell(0)| + \gamma \eta (1 + |x|),$

and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t, x)$:

$$\lim_{n \to \infty} v^*(t_n, x_n, p_n) = \lim_{n \to \infty} v^*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty,$$

$$\lim_{n \to \infty} v^*(t_n, x_n, p_n) = \lim_{n \to \infty} v^*(t_n, x_n, p_n) = 1 D\ell(\infty) \text{ if } p_n \to \infty.$$

A little more: $v$ is continuous in $p$ and $x$. 
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$$0 \leq \nu(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta (1 + |x|),$$
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More controls on $\nu$

□ It also implies that $\exists \eta > 0$ s.t.

$$0 \leq \nu(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta(1 + |x|),$$

□ and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t, x)$:

$$\lim_{n \to \infty} \nu_*(t_n, x_n, p_n) = \lim_{n \to \infty} \nu_*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty,$$

$$\lim_{n \to \infty} \frac{\nu_*(t_n, x_n, p_n)}{p_n} = \lim_{n \to \infty} \frac{\nu_*(t_n, x_n, p_n)}{p_n} = \frac{1}{D\ell(\infty)} \text{ if } p_n \to \infty.$$

□ A little more: $\nu$ is continuous in $p$ and $x^3$. 
Want a comparison result in the class of function with the above limit and growth conditions.
Uniqueness

- Want a comparison result in the class of functions with the above limit and growth conditions.

- Recall that

\[ F\varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi/D_p \varphi|^2 D_p^2 \varphi - 2 (D_{x^1} \varphi/D_p \varphi) D_{(x^1,p)}^2 \varphi \right) . \]
Uniqueness

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- We now control $D_{x^1} \varphi / D_p \varphi$. 

- Bound on the stock price...
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- We now control \( D_{x^1} \varphi / D_p \varphi \).

This is not enough... If we need to penalize in \( x^1 \) (stock price) then the term \( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi \) will blow up as \( n \to \infty \), where \( n \) comes from the usual penalisation \( n|x_1^1 - x_2^1|^2 \) due to the doubling of constants.
Uniqueness

- Want a comparison result in the class of function with the above limit and growth conditions.

- Recall that

\[ F_\varphi := \frac{\sigma^2(x^1)}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1,p)}^2 \varphi \right). \]

- We now control \( D_{x^1} \varphi / D_p \varphi \).

Assumption:

\[ \exists \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1). \]
Uniqueness

- Want a comparison result in the class of function with the above limit and growth conditions.

- Recall that

\[ F_\varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( \left| \frac{D_{x^1} \varphi}{D_p \varphi} \right|^2 D_p^2 \varphi - 2 \left( \frac{D_{x^1} \varphi}{D_p \varphi} \right) D^2_{(x^1,p)} \varphi \right) \].

- We now control \( \frac{D_{x^1} \varphi}{D_p \varphi} \).

Assumption:

\[ \exists \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1) \].

- Bound on the stock price...
\textbf{Theorem} : Let $U$ (resp. $V$) be a non-negative super- and subsolutions which are continuous in $x^3$. Assume that

$$U(t, x, p) \geq V(t, x, p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and that $\exists \ c_+ > 0$ and $c_- \in \mathbb{R}$ s.t.

$$\limsup_{(t', x', p') \to (t, x, \infty)} V(t', x', p')/p' \leq c_+ \leq \liminf_{(t', y', p') \to (t, y, \infty)} U(t', y', p')/p',$$

$$\limsup_{(t', x', p') \to (t, x, -\infty)} V(t', x', p') \leq c_- \leq \liminf_{(t', y', p') \to (t, y, -\infty)} U(t', y', p').$$

If either $U$ is a supersolution of (*) which is continuous in $p$, or $V$ is a subsolution of (**) which is continuous in $p$, then

$$U \geq V.$$
Additional remarks
Serves as a building block for problems of the form

$$\sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E} \left[ U(X_{t,x}^\phi(T), Y_{t,z}^\phi(T)) \right]$$

with \( \mathcal{A}_{t,z} := \{ \phi \in \mathcal{A} : Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \} \) .
Optimal management under shortfall constraints

Serves as a building block for problems of the form

$$
\sup_{\phi \in A_{t,z}} \mathbb{E} \left[ U(X_{t,x}^\phi(T), Y_{t,z}^\phi(T)) \right]
$$

with $A_{t,z} := \{ \phi \in A : Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \}$.

Amongs to say that $Y_{t,z}^\phi \geq v(\cdot, X_{t,x}^\phi)$

where $v(t, x) := \inf \left\{ y : \exists \phi \in A \text{ s.t. } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \right\}$,

see B., Elie and Imbert (2010).
Consider the control problem:

\[ w := \inf_{\phi} \mathbb{E}\left[ U(X^{\phi}(T)) \right] \]

Allows for a unified approach (obviously obtains immediately the same HJB PDE)

Optimal control vs stochastic targets

Consider the control problem:

\[ w := \inf_{\phi} \mathbb{E} \left[ U(X_\phi(T)) \right] \]

Then, it can be written as a stochastic target problem

\[ w = v := \inf \left\{ p : \exists (\phi, \alpha) \text{ s.t. } U(X_\phi(T)) \leq P^\alpha_p(T) \right\} \]

with \( P^\alpha_p := p + \int_0^T \alpha_s dW_s \).
Consider the control problem:

\[ w := \inf_{\phi} \mathbb{E} \left[ U(X^\phi(T)) \right] \]

Then, it can be written as a stochastic target problem

\[ w = v := \inf \left\{ p : \exists (\phi, \alpha) \text{ s.t. } U(X^\phi(T)) \leq P_p^\alpha(T) \right\} \]

with \( P_p^\alpha := p + \int_0^T \alpha_s dW_s \).

Allows for a unified approach (obviously obtains -immediately- the same HJB PDE)

Game version: parameter uncertainty

- Problem:

\[ \nu := \inf \left\{ y : \exists \phi \text{ s.t. } \mathbb{E} \left[ G(X^{\phi[\vartheta]}, Y^{\phi[\vartheta]}(T)) \right] \geq p \ \forall \vartheta \right\} . \]

- Examples: Unknown volatility, drift, market volume, ...

- B., Moreau and Nutz (in -good- progress).