A class of domains with fractal boundaries: Functions spaces and numerical methods

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Geometrical construction
Function spaces on $\Gamma$ and known results
New trace and extension results
Numerical methods
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A class of self-similar sets $\Gamma$

Define two similitudes $f_1$ and $f_2$
with ratio $a < 1$ and opposite
angles $\pm \theta$ ($0 \leq \theta < \frac{\pi}{2}$).
A class of self-similar sets $\Gamma$

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The set $\Gamma$

$\Gamma$ is said to be the self-similar set associated to the similitudes $f_1$ and $f_2$, i.e. the unique compact set in $\mathbb{R}^2$ such that

$$\Gamma = f_1(\Gamma) \cup f_2(\Gamma).$$
A class of self-similar sets $\Gamma$

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$$\Gamma = f_1(\Gamma) \cup f_2(\Gamma).$$

It can be proved that the Hausdorff dimension of $\Gamma$ is

$$d := \dim_H(\Gamma) = -\frac{\log 2}{\log a}.$$
The self-similar measure on $\Gamma$

**Theorem (J. E. Hutchinson, 1981)**

There exists a unique Borel invariant probability measure $\mu$ on the self-similar set $\Gamma$, in the sense that for every Borel set $B$ in $\Gamma$,

$$
\mu(B) = \mu(f_1^{-1}(B)) + \mu(f_2^{-1}(B)).
$$

The critical value $a^*$

There exists a critical value $a^*$ depending on $\theta$ such that:

- if $a < a^*$, then $\Gamma$ is totally disconnected,

\[ a < a^* \]
The critical value $a^*$

There exists a critical value $a^*$ depending on $\theta$ such that:

- if $a < a^*$, then $\Gamma$ is totally disconnected,
- if $a = a^*$, then $\Gamma$ is connected and has multiple points.
A class of ramified domains (1/2)
A class of ramified domains (1/2)

\[ f_1(\Gamma^0) \]

---

A class of self-similar sets
A class of ramified domains

Y. Achdou
Sobolev extension property
A class of ramified domains (1/2)

\[ f_1(\Gamma^0) \quad \text{and} \quad f_2(\Gamma^0) \]

\[ \Gamma^0 \]
A class of ramified domains (1/2)
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A class of ramified domains (1/2)
A class of ramified domains (2/2)

If $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathcal{A}_n$, \( \mathcal{A}_n := \{1, 2\}^n \), write

$$f_\sigma = f_{\sigma(1)} \circ \cdots \circ f_{\sigma(n)}.$$

Define

$$\Omega = \text{Interior} \left( \bigcup_{n \in \mathbb{N}} \bigcup_{\sigma \in \mathcal{A}_n} f_\sigma(Y^0) \right).$$
Function spaces on closed sets

**Definition (d-set)**

If $F$ is a closed set in $\mathbb{R}^n$ and $0 < d \leq n$, a Borel measure $m$ on $F$ is said to be a $d$-measure if there exist constants $c_1, c_2 > 0$ s.t.

$$\forall x \in F, \forall r \in (0, 1), \quad c_1 r^d \leq m(B(x, r)) \leq c_2 r^d,$$

where $B(x, r)$ is the euclidean metric ball. A closed set having a $d$-measure is referred to as a $d$-set.
Function spaces on closed sets

Definition ($d$-set)

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where $B(x, r)$ is the euclidean metric ball. A closed set having a $d$-measure is referred to as a $d$-set.

Definition (Sobolev spaces on $d$-sets)

For $0 < s < 1$ and $1 \leq p \leq \infty$, if $v \in L^p_m(F)$ then $v \in W^{s,p}(F)$ iff

$$|v|_{W^{s,p}(F)} := \iint_{|x-y|<1} \frac{|v(x) - v(y)|^p}{|x-y|^{d+ps}} \ d m(x) d m(y) < \infty.$$
A trace theorem on \(d\)-sets

**Theorem (A. Jonsson, H. Wallin, 1984)**

If \(F \subset \mathbb{R}^n\) is a \(d\)-set, \(0 < d < n\), \(1 < p < \infty\), and \(1 - \frac{n-d}{p} > 0\), then

\[
W^{1,p}(\mathbb{R}^n)_{\mid F} = W^{1-\frac{n-d}{p},p}(F).
\]
A trace theorem on \( d \)-sets

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\[
W^{1,p}(\mathbb{R}^n)|_F = W^{1-\frac{n-d}{p},p}(F).
\]

**Definition of the trace:** If \( u \in L^1_{loc}(\mathbb{R}^n) \) then \( u \) is strictly defined at \( x \in \mathbb{R}^n \) if

\[
\bar{u}(x) := \lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y) \, dy
\]

exists. We define the trace of \( u \) on \( F \) to be \( \bar{u}|_F \), defined only on those points where \( u \) is strictly defined.
A trace theorem on \(d\)-sets

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If \(F \subset \mathbb{R}^n\) is a \(d\)-set, \(0 < d < n\), \(1 < p < \infty\), and \(1 - \frac{n-d}{p} > 0\), then

\[
W^{1,p}(\mathbb{R}^n)|_F = W^{1-\frac{n-d}{p},p}(F).
\]

**Examples**

- \(d = n - 1\), \(p = 2\):
  \[
  W^{1,2}(\mathbb{R}^n)|_F = W^{1/2,2}(F) = H^{1/2}(F).
  \]

- \(n = 2\), \(F = \) Cantor set in a line segment, \(d = \log 2 / \log 3\),
  \[
  W^{1,2}(\mathbb{R}^2)|_F = W^{d/2,2}(F).
  \]
Consequence: trace results on $\Gamma$

**Proposition**

The set $\Gamma$ endowed with the self-similar measure $\mu$ is a $d$-set, where $d = -\frac{\log 2}{\log a}$.

**Corollary**

If $p > 1$, then

$$W^{1,p}(\mathbb{R}^2)|_\Gamma = W^{1-\frac{2-d}{p},p}(\Gamma).$$

**Question:** can we characterize the trace on $\Gamma$ of $W^{1,p}(\Omega)$?
Extension domains

**Definition ($W^{k,p}$-extension domain)**

A domain $D \subset \mathbb{R}^n$ is called a $W^{k,p}$-extension domain ($k \in \mathbb{N}$, $1 \leq p \leq \infty$) if there exists a continuous linear extension operator

$$\Lambda : W^{k,p}(D) \rightarrow W^{k,p}(\mathbb{R}^n).$$
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If $D$ is a $W^{k,p}$-extension domain for every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, then $D$ is said to be a Sobolev extension domain.
**Extension domains**

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A domain \(D \subset \mathbb{R}^n\) is called a \(W^{k,p}\)-extension domain (\(k \in \mathbb{N}, 1 \leq p \leq \infty\)) if there exists a continuous linear extension operator

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**Definition (Sobolev extension domain)**

If \(D\) is a \(W^{k,p}\)-extension domain for every \(k \in \mathbb{N}\) and \(1 \leq p \leq \infty\), then \(D\) is said to be a Sobolev extension domain.

**Theorem (A. P. Calderón, E. M. Stein, 1970)**

*Every Lipschitz domain is a Sobolev extension domain.*
(\varepsilon, \delta)\text{-domains}

**Definition ((\varepsilon, \delta)\text{-domain})**

A domain $D$ in $\mathbb{R}^n$ is said to be an $((\varepsilon, \delta)\text{-domain})$ if there exist $\varepsilon, \delta > 0$ such that for every $x, y \in D$ satisfying $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset D$ joining $x$ and $y$ such that:

- $l(\gamma) \leq \frac{1}{\varepsilon} |x - y|,$
- $d(z, D^c) \geq \varepsilon \frac{|x - z| \cdot |y - z|}{|x - y|},$ for all $z \in \gamma.$
An Example

The Koch flake is an \((\varepsilon, \delta)\)-domain.
Extension theorems

Theorem (P. W. Jones, 1974)

*Every* $(\varepsilon, \delta)$-*domain in* $\mathbb{R}^n$ *is a Sobolev extension domain.*
Extension theorems

Theorem (P. W. Jones, 1974)

Every $(\varepsilon, \delta)$-domain in $\mathbb{R}^n$ is a Sobolev extension domain.

The result is almost sharp in $\mathbb{R}^2$:

Theorem (P. W. Jones, 1974)

In $\mathbb{R}^2$, if a domain $D$ is finitely connected, then $D$ is a Sobolev extension domain if and only if $D$ is an $(\varepsilon, \delta)$-domain.
The subcritical case

In the subcritical case where \( a < a^* \), the set \( \Gamma \) is totally disconnected, and the ramified domain \( \Omega \) is an \((\varepsilon, \delta)\)-domain.

Hence, \( \Omega \) is a Sobolev extension domain, and the trace result holds for all \( p \in (1, \infty) \):

\[
W^{1,p}(\Omega)_{|\Gamma} = W^{1-\frac{n-d}{p},p}(\Gamma).
\]
The critical case

In the critical case where $a = a^*$, the set $\Gamma$ is connected. In this case, $\Omega$ is not an $(\varepsilon, \delta)$-domain.

In this case, $\Omega$ is not a $W^{1,p}$-extension domain for $p > 2$. The trace space $W^{1,p}(\Omega)|_{\Gamma}$ cannot be easily characterized.
Questions raised by the previous analysis

In the critical case, we shall see that the traces of functions of $W^{1,p}(\Omega)$ on $\Gamma$ cannot always be characterized as Sobolev spaces.
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The critical case: questions

- characterize the trace space of $W^{1,p}(\Omega)$ on $\Gamma$: new function spaces?
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In the critical case, we shall see that the traces of functions of \( W^{1,p}(\Omega) \) on \( \Gamma \) cannot always be characterized as Sobolev spaces.

The critical case: questions

- Characterize the trace space of \( W^{1,p}(\Omega) \) on \( \Gamma \): new function spaces?
- Find relations between the trace spaces and Sobolev spaces if possible: the dimension of the self-intersection will play a role
Questions raised by the previous analysis

In the critical case, we shall see that the traces of functions of $W^{1,p}(\Omega)$ on $\Gamma$ cannot always be characterized as Sobolev spaces.

The critical case: questions

- characterize the trace space of $W^{1,p}(\Omega)$ on $\Gamma$: new function spaces?
- Find relations between the trace spaces and Sobolev spaces if possible: the dimension of the self-intersection will play a role
- Extension results
Construction of a trace operator $\Omega \rightarrow \Gamma$: a sequence of operators $\ell^n : W^{1,p}(\Omega) \rightarrow L^p_\mu(\Gamma)$
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The trace operator on $\Gamma$

Consider the sequence of operators $\ell^n : W^{1,p}(\Omega) \to L^p_\mu(\Gamma)$ defined by

$$\ell^n(u) = \sum_{\sigma \in A_n} \langle u \rangle_{\Gamma^\sigma} \mathbb{1}_{f_\sigma(\Gamma)},$$

where $\langle u \rangle_{\Gamma^\sigma} = \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u$ and $\Gamma^\sigma = f_\sigma(\Gamma)$. 
The trace operator on $\Gamma$

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$$

where $\langle u \rangle_{\Gamma^\sigma} = \frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} u$ and $\Gamma^\sigma = f_\sigma(\Gamma)$.

**Proposition**

The sequence $(\ell^n)_n$ converges in $\mathcal{L}(W^{1,p}(\Omega), L^p_\mu(\Gamma))$ to a continuous operator $\ell^\infty$. 

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Sobolev extension property
Haar wavelets on $\Gamma$

We define the Haar wavelets on $\Gamma$ by

\[
\begin{align*}
 g_0 &= \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\
g_\sigma|_{f_\sigma(\Gamma)} &= 2^{\frac{n}{2}} g_0 \circ f_\sigma^{-1} \quad \text{and} \quad g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \quad \text{for} \quad \sigma \in \mathcal{A}_n
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Mother wavelet $g_0$
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$$

Mother wavelet $g_0$ \hspace{1cm} $g_\sigma$ for $\sigma = (1)$
Haar wavelets on $\Gamma$

We define the Haar wavelets on $\Gamma$ by

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\begin{align*}
g_0 &= \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\
g_\sigma|_{f_\sigma(\Gamma)} &= 2^{\frac{n}{2}} g_0 \circ f_\sigma^{-1} \text{ and } g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \text{ for } \sigma \in \mathcal{A}_n
\end{align*}
\]

Mother wavelet $g_0$

$g_\sigma$ for $\sigma = (1)$

$g_\sigma$ for $\sigma = (12)$
Haar wavelets on $\Gamma$

We define the Haar wavelets on $\Gamma$ by

$$
\begin{align*}
g_0 & = 1_{f_1(\Gamma)} - 1_{f_2(\Gamma)} \\
g_\sigma|_{f_\sigma(\Gamma)} & = 2^{\frac{n}{2}} g_0 \circ f_\sigma^{-1} \quad \text{and} \quad g_\sigma|_{\Gamma \setminus f_\sigma(\Gamma)} = 0 \quad \text{for} \quad \sigma \in \mathcal{A}_n
\end{align*}
$$

Mother wavelet $g_0$ \quad $g_\sigma$ for $\sigma = (1)$ \quad $g_\sigma$ for $\sigma = (12)$

Every function $f \in L_p^p(\Gamma)$, $1 \leq p < \infty$ can be expanded in the Haar wavelet basis $\{g_\sigma, \sigma \in \mathcal{A}\}$:

$$
f = \langle f \rangle_{\Gamma} + \sum_{n \geq 0} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma g_\sigma.
$$
We distinguish two different cases.
Call $\Xi$ the self-intersection set of $\Gamma$.

\[ \theta \not\in \{ \frac{\pi}{2k}, \ k \in \mathbb{N}^* \}, \]

$\Xi$ is countable,
We distinguish two different cases.

Call $\Xi$ the self-intersection set of $\Gamma$.

$$\theta \notin \{ \frac{\pi}{2k}, \, k \in \mathbb{N}^* \},$$

$\Xi$ is countable,

$$\theta = \frac{\pi}{2k}, \, k \in \mathbb{N}^*,$$

$$\dim_H \Xi = \frac{d}{2}.$$
Regularities of the Haar wavelets

Proposition \((\dim_{H}(\Xi) = 0)\)

*In the case when* \(\theta \neq \frac{\pi}{2k}\), \(k \in \mathbb{N}^*\), \(g_0 \in W^{s,p}(\Gamma)\) \(\text{if } s < \frac{d}{p}\), \(g_0 \notin W^{s,p}(\Gamma)\) \(\text{if } s > \frac{d}{p}\).
Regularity of the Haar wavelets

**Proposition (dim$_H(\Xi) = 0)$**

In the case when $\theta \neq \frac{\pi}{2k}$, $k \in \mathbb{N}^*$,

\begin{align*}
g_0 &\in W^{s, p}(\Gamma) \quad \text{if } s < \frac{d}{p}, \\
g_0 &\notin W^{s, p}(\Gamma) \quad \text{if } s > \frac{d}{p}.
\end{align*}

**Proposition (dim$_H(\Xi) = d/2)$**

In the case when $\theta = \frac{\pi}{2k}$, $k \in \mathbb{N}^*$,

\begin{align*}
g_0 &\in W^{s, p}(\Gamma) \quad \text{if } s < \frac{d}{2p}, \\
g_0 &\notin W^{s, p}(\Gamma) \quad \text{if } s > \frac{d}{2p}.
\end{align*}
New function spaces

**Definition (\(JLip\) spaces, A. Jonsson 2004)**

The space \(JLip^s;p(\Gamma)\) consists of all the functions \(f \in L^p_\mu(\Gamma)\) s.t.

\[
\sum_{n \geq 0} 2^{ns} \frac{p}{d} 2^n \left(\frac{p}{2} - 1\right) \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p < \infty,
\]

where \(\beta_\sigma\) are the Haar wavelet coefficients of \(f\).

**Equivalent definition**

\[
\|f\|_{L^p_\mu}^p + \sum_{n=0}^\infty 2^{ns} \frac{p}{d} \sum_{\sigma \in \mathcal{A}_n} \int_{\Gamma^\sigma} |f - \langle f \rangle f_\sigma(\Gamma)|^p d\mu < \infty.
\]
A characterization of the trace space on $\Gamma$

**Theorem (Y.A., N. Tchou, 2010)**

If $1 \leq p < \infty$, then

$$\ell^\infty(W^{1,p}(\Omega)) = JLip^{1-\frac{2-d}{p},p}(\Gamma).$$
**Idea of the proof**

\[ J Lip^{1-\frac{2-d}{p}p}(\Gamma) \subset \ell^\infty(W^{1,p}(\Omega))? \]

Explicit construction of a lifting \( E : J Lip^{1-\frac{2-d}{p}p}(\Gamma) \to W^{1,p}(\Omega) \), by using the expansion of a funct. \( f \in J Lip^{1-\frac{2-d}{p}p}(\Gamma) \) on Haar wavelets.

\[ \ell^\infty(W^{1,p}(\Omega)) \subset J Lip^{1-\frac{2-d}{p}p}(\Gamma)? \]

- Prove the strengthened trace inequality: \( \forall \rho \in (2^{(p-1)(1-\frac{2}{d})}, 1) \), \( \exists C \) s.t. \( \forall v \in W^{1,p}(\Omega) \),

\[
\| \ell^\infty(v) - \langle \ell^\infty(v) \rangle_\Gamma \|_{L^p_{\mu}}^p \\
\leq C \left( \| \nabla v \|_{L^p(Y_0)}^p + \sum_{n=1}^{\infty} \rho^{-n} 2^{n(p-1)(1-\frac{2}{d})} \sum_{\sigma \in A_n} \| \nabla v \|_{L^p(f_\sigma(Y_0))}^p \right).
\]

- Use self-similarity to conclude.
For the strengthened trace inequality, map the domain $\Omega$ to a fractured domain $\hat{\Omega}$ with vertical boundaries (Weierstrass curve):
Relationship with classical Sobolev spaces

Theorem (Y.A., T. Deheuvels, N. Tchou, 2010)

1. If $0 < t < \min\left(\frac{d-\dim_H(\Xi)}{p}, 1\right)$, then $JLip^t,p(\Gamma) = W^{t,p}(\Gamma)$,

2. For all $t \in (0, 1)$ and $s > \frac{d-\dim_H(\Xi)}{p}$, $JLip^t,p(\Gamma) \not\subset W^{s,p}(\Gamma)$.

(recall that $\dim_H(\Xi) = 0$ or $\dim_H(\Xi) = d/2$)
Relationship with classical Sobolev spaces

Theorem (Y.A., T. Deheuvels, N. Tchou, 2010)

1. If $0 < t < \min\left(\frac{d-\dim_H(\Xi)}{p}, 1\right)$, then $JLip_t^{t,p}(\Gamma) = W_t^{t,p}(\Gamma)$.
2. For all $t \in (0,1)$ and $s > \frac{d-\dim_H(\Xi)}{p}$, $JLip^{t,p}(\Gamma) \not\subset W^{s,p}(\Gamma)$.

(recall that $\dim_H(\Xi) = 0$ or $\dim_H(\Xi) = d/2$)

Corollary (Y.A., T. Deheuvels, N. Tchou, 2010)

Define $p^* \equiv 2 - \dim_H(\Xi)$.

1. If $1 \leq p < p^*$, then $\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma)$.
2. If $p \geq p^*$, then $\ell^\infty(W^{1,p}(\Omega)) \subset W^{s,p}(\Gamma)$ for all $s < \frac{d-\dim_H(\Xi)}{p}$

The regularity of the wavelets shows that the result is sharp.
Principle of the proof

Estimate the $W^{s,p}(\Gamma)$-norm of a function $u \in JLip^{t,p}$ by using

- a partition of $\Gamma$ and all its images by the similarities $f_\sigma$
- the Haar wavelet decomposition of $u$
- discrete Hardy inequalities: $\forall \gamma \in \mathbb{R}, \forall p \geq 1, \exists C$ s.t., for any sequence of positive real numbers $(c_k)_{k \in \mathbb{N}},$

$$\sum_{n \in \mathbb{N}} 2^{\gamma n} \left( \sum_{k \leq n} c_k \right)^p \leq C \sum_{n \in \mathbb{N}} 2^{\gamma n} c_n^p \quad \text{if } \gamma < 0,$$

$$\sum_{n \in \mathbb{N}} 2^{\gamma n} \left( \sum_{k \geq n} c_k \right)^p \leq C \sum_{n \in \mathbb{N}} 2^{\gamma n} c_n^p \quad \text{if } \gamma > 0.$$
$W^{1,p}$-Extension property of $\Omega$

**Theorem (T. Deheuvels, 2011)**

*It is possible to construct a linear extension operator $P$, which is continuous from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$ for all $p < p^*$.*

The following theorem *a posteriori* justifies the use of several notions of trace on $\Gamma$.

**Theorem (Y.A., T. Deheuvels, N. Tchou, 2011)**

*For all $p \geq 1$, every function $u \in W^{1,p}(\Omega)$ is strictly defined $\mu$-almost everywhere on $\Gamma$, and*

$$
\tilde{u}|_{\Gamma} = \ell^\infty(u), \quad \mu\text{-almost everywhere on } \Gamma
$$

The proof uses the extension operator $P$ constructed for $p < p^*$. 
Corollary (Y.A., T. Deheuvels, N. Tchou, 2011)

With $p^*$ defined by $p^* = 2 - \dim_H(\Xi)$,

- for all $p < p^*$, $\Omega$ is a $W^{1,p}$-extension domain,
- for all $p > p^*$, $\Omega$ is not a $W^{1,p}$-extension domain.
Boundary problems with Neumann conditions on \( \Gamma \)

For \( g \in L^2_\mu(\Gamma) \) and \( u \in H^{1/2}(\Gamma^0) \), \( \exists! \ w \in H^1(\Omega) \) s.t.

\[
w|_{\Gamma^0} = u, \quad \text{and} \quad \int_\Omega \nabla w \cdot \nabla v = \int_\Gamma g l^\infty(v) d\mu, \quad \forall v \in \mathcal{V}(\Omega). \quad (*)
\]

where

\[
\mathcal{V}(\Omega) = \{ v \in H^1(\Omega) \text{ s.t. } v|_{\Gamma^0} = 0 \}.
\]

Goal

Let \( Z^n \) be the ramified domain truncated after the \( n \)-th generation.

The goal is to compute \( w|_{Z^n} \) with no error due to domain truncation, by using transparent boundary conditions.
A Dirichlet to Neumann operator

**Theorem (Harmonic lifting: energy decay)**

Call $\mathcal{H}(u)$ the solution to (*) with $g = 0$. There exists a constant $\rho < 1$ s.t.
for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,

$$
\int_{\Omega \setminus \mathbb{Z}^p} |\nabla \mathcal{H}(u)|^2 \leq \rho^p \int_{\Omega} |\nabla \mathcal{H}(u)|^2, \quad \forall p \geq 0.
$$

**Definition (Dirichlet to Neumann operator)**

Let $T$ be the Dirichlet to Neumann operator:

$$
Tu = \frac{\partial \mathcal{H}(u)}{\partial n} |_{\Gamma^0} \iff \langle Tu, v \rangle = \int_{\Omega} \nabla \mathcal{H}(u) \cdot \nabla \mathcal{H}(v).
$$
Transparent boundary conditions

Main idea

Self-similarity and scale invariance of the PDE imply that:
if \( w = \mathcal{H}(u) \) then

\[
\begin{align*}
\left. w \right|_{f_1(\Omega)} \circ f_1 &= \mathcal{H} \left( \left. w \right|_{f_1(\Gamma^0)} \circ f_1 \right), \\
\left. w \right|_{f_2(\Omega)} \circ f_2 &= \mathcal{H} \left( \left. w \right|_{f_2(\Gamma^0)} \circ f_2 \right).
\end{align*}
\]
Main idea

Self-similarity and scale invariance of the PDE imply that:
if \( w = \mathcal{H}(u) \) then

\[
\begin{align*}
  w|_{f_1(\Omega)} \circ f_1 &= \mathcal{H} \left( w|_{f_1(\Gamma^0)} \circ f_1 \right), \\
  w|_{f_2(\Omega)} \circ f_2 &= \mathcal{H} \left( w|_{f_2(\Gamma^0)} \circ f_2 \right).
\end{align*}
\]

If \( T \) is available, then \( w|_{Y^0} \) can be computed by solving a boundary value problem in \( Y^0 \).
Main idea

Self-similarity and scale invariance of the PDE imply that:
if \( w = \mathcal{H}(u) \) then

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\begin{align*}
    w|_{f_1(\Omega)} \circ f_1 &= \mathcal{H} (w|_{f_1(\Gamma^0)} \circ f_1), \\
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If \( T \) is available, then \( w|_{Y^0} \) can be computed by solving a boundary value problem in \( Y^0 \).

\[ \Delta w = 0 \quad \text{and} \quad w|_{\Gamma^0} = u \quad \text{(and} \quad \frac{\partial w}{\partial n} - Tu = 0). \]
Main idea

Self-similarity and scale invariance of the PDE imply that:
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If \( T \) is available, then \( w|_{Y^0} \) can be computed by solving a boundary value problem in \( Y^0 \).

\[
\begin{align*}
    \frac{\partial w}{\partial n} \circ f_1 + \frac{1}{\alpha} T(w|_{f_1(\Gamma^0)} \circ f_1) &= 0, \\
    \frac{\partial w}{\partial n} \circ f_2 + \frac{1}{\alpha} T(w|_{f_2(\Gamma^0)} \circ f_2) &= 0, \\
    \Delta w &= 0, \\
    w|_{\Gamma^0} &= u \text{ (and } \frac{\partial w}{\partial n} - Tu = 0) \end{align*}
\]
Solution in $Z^n$ with arbitrary small truncation error

**Algorithm**

If $T$ is available, then for all $n \geq 0$, one can find $\mathcal{H}(u)|_{Z^n}$ by solving sequentially $1 + 2 + \cdots + 2^{n-1}$ boundary value problems in $Y^0$ with nonlocal transparent conditions on $f_1(\Gamma^0) \cup f_2(\Gamma^0)$ involving $T$, and Dirichlet data on $\Gamma^0$ computed from the previous step.
Fixed point iterations for computing $T$

Let $\mathcal{O}$ be the cone of self-adjoint, positive semi-definite, continuous linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, vanishing on the constants. Clearly $T \in \mathcal{O}$.

Take the map $\mathbb{M} : S \in \mathcal{O} \mapsto \mathbb{M}(S) \in \mathcal{O}$, where $\mathbb{M}(S)u = \frac{\partial w}{\partial n}|_{\Gamma^0}$, with

$$
\begin{cases}
\Delta w = 0 & \text{in } Y^0, \\
\frac{\partial w}{\partial n}|_{f_i(\Gamma^0)} \circ f_i = -\frac{1}{a} S(w|_{f_i(\Gamma^0)} \circ f_i) & i = 1, 2 \\
\frac{\partial w}{\partial n} = 0 & \text{on the rest of the boundary of } Y^0
\end{cases}
$$

Theorem

The operator $T$ is the unique fixed point of $\mathbb{M}$. Moreover, for all $S \in \mathcal{O}$,

$$
||\mathbb{M}^p(S) - T|| \leq C \rho^p.
$$
The case when $g \neq 0$

The following strategy may be used for computing $w|_{Z^n}$ with an arbitrary accuracy:

- If $g$ is Haar wavelet then from self-similarity, $w|_{Z^n}$ can be computed exactly by induction w.r.t. the wavelets, by solving a finite number of boundary value problems in $Y^0$ with nonhomogeneous transparent boundary conditions on $f_1(\Gamma^0) \cup f_2(\Gamma^0)$ involving the transparent condition operator $T$. 
The case when $g \neq 0$

The following strategy may be used for computing $w|_{Z^n}$ with an arbitrary accuracy:

- If $g$ is Haar wavelet then from self-similarity, $w|_{Z^n}$ can be computed exactly by induction w.r.t. the wavelets, by solving a finite number of boundary value problems in $Y^0$ with nonhomogeneous transparent boundary conditions on $f_1(\Gamma^0) \cup f_2(\Gamma^0)$ involving the transparent condition operator $T$.

- For a general $g$, expand $g$ on the basis of Haar wavelets, and use the linearity of the problem.
The error may be made as small as desired

We obtain a method for approximating $w|_{Z^n}$ and error estimates (depending on the truncation in the wavelet expansion and of the approximation of $T$)

**Theorem**

There exists $\rho < 1$ and a constant $C$ independent of $g$ s.t.

$\forall n, P \in \mathbb{N}$ with $1 \leq n < P$

$$\|error^{(P)}\|_{H^1(Z^{n-1})} \leq C \sqrt{2-P} \rho^{P-n} \|g\|_{L^2_{\mu}},$$

where the wavelet expansion of $g$ is truncated at level $P$. 
The Neumann datum is a Haar wavelet
The Neumann datum is $\cos\left(\frac{3\pi s}{2}\right)g_0(s)$

Expansion with wavelets of levels $\leq 5$ (resp. $\leq 4$, $\leq 2$)

Error in $L_2$ norm (logscale) in $Y^0, Z^1, \ldots, Z^5$ w.r.t. the truncation of the wavelet expansion
Helmholtz equation $\Delta u + k^2 u = 0$

The operator is no longer scale invariant. Therefore, a whole sequence of operators $T_{a^{2p}k}$ is needed. But there is a backward induction relation:

$$T_{a^{2p}k} \rightarrow T_{a^{2(p-1)}k} \rightarrow \cdots \rightarrow T_{a^{4}k} \rightarrow T_{a^{2}k} \rightarrow T_{k}.$$  

and

$$\lim_{p \to \infty} T_{a^{2p}k} = T_{0}.$$  

This observation allows one for designing an algorithm to solve the Helmholtz equation with an arbitrary small truncation error, except maybe for special frequencies (the eigenfrequencies of the pb).
Example: eigenmodes of the Laplace operator with Neumann cond. on $\Gamma^0$

Left: the third eigenmode, restricted to $Z^3$. Right: the sixth eigenmode, restricted to $Z^3$. 
Linear evolution equations (e.g. the wave equation)

Strategy

- Compute a large number of (normalized) eigenmodes
- Compute the expansion of solution of the wave equation on the eigenmodes.

The solution of the wave equation vs. $t$ at a given point of $\Omega$, with a compactly supported $u_0$ and $u_1 = 0$