Functional CLT for Linear Processes with Long Memory

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Plan of talk

- Central limit theorem for linear processes.
- Functional central limit theorem for linear processes.
- Selfnormalized CLT.
- Exact asymptotic for linear processes
CLT for linear processes with finite second moments

\[ X_k = \sum_{j=\infty}^{\infty} a_{k+j} \xi_j, \quad S_n = S_n(X) = \sum_{j=1}^{n} X_j, \]

**Theorem**

(Ibragimov and Linnik, 1971) Let \((\xi_j)\) be i.i.d. centered with finite second moment, \(\sum_{k=-\infty}^{\infty} a_k^2 < \infty\) and \(\sigma_n^2 = \text{var}(S_n) \to \infty\). Then

\[ S_n / \sigma_n \xrightarrow{D} N(0, 1). \]

\[ \sigma_n^2 = \sum_{j=-\infty}^{\infty} b_{nj}^2, \quad b_{nj} = a_{j+1} + \ldots + a_{j+n}. \]

It was conjectured that a similar result might hold without the assumption of finite second moment.
CLT for linear processes with infinite second moments

\((\ast)\quad H(x) = \mathbb{E}(\xi_0^2 I(\left|\xi_0\right| \leq x))\) is a slowly varying function at \(\infty\).

\(X_0\) is well defined if

\[
\sum_{j \in \mathbb{Z}, a_j \neq 0} a_j^2 H(|a_j|^{-1}) < \infty,
\]

**Theorem**

(P-Sang, 2011) Let \((\xi_k)_{k \in \mathbb{Z}}\) be i.i.d, centered. Then the following statements are equivalent:

1. \(\xi_0\) is in the domain of attraction of the normal law (i.e. satisfies \((\ast)\))
2. For any sequence of constants \((a_n)_{n \in \mathbb{Z}}\) as above and \(\sum_{j=\infty}^{\infty} b_{nj}^2 \to \infty\) the CLT holds. (i.e. \(S_n / D_n \to N(0, 1)\))

\[
D_n = \inf \left\{ s \geq 1 : \sum \frac{b_{n,k}^2}{s^2} H \left( \frac{s}{|b_{n,k}|} \right) \leq 1 \right\}, \text{ and } D_n^2 \sim \sum b_{n,k}^2 \xi_k^2.
\]
For $0 \leq t \leq 1$ define

$$W_n(t) = \frac{\sum_{i=1}^{[nt]} X_i}{\sigma_n}$$

where $[x]$ is the integer part of $x$.

**Problem**

Let $(\xi_j)$ be i.i.d. centered with finite second moment, $\sum_{k=-\infty}^{\infty} a_k^2 < \infty$ and $\sigma_n^2 = nh(n)$ with $h(x)$ a function slowly varying at $\infty$. Is it true that $W_n(t) \Rightarrow W(t)$, where $W(t)$ is the standard Brownian motion?

This will necessarily imply in particular that for every $\varepsilon \geq 0$,

$$\mathbb{P}(\max_{1 \leq i \leq n} |X_i| \geq \varepsilon \sigma_n) \to 0 \text{ as } n \to \infty.$$
Example

There is a linear process \((X_k)\) such that \(\sigma_n^2 = nh(n)\) and such that the weak invariance principle does not hold:

\[
\mathbb{P}(\|\tilde{\xi}_0\| > x) \sim \frac{1}{x^2 \log^{3/2} x},
\]

\[a_0 = 0, \quad a_1 = \frac{1}{\log 2} \quad \text{and} \quad a_n = \frac{1}{\log(n + 1)} - \frac{1}{\log n}, \quad \text{for} \quad n \geq 2,
\]

\[
\sigma_n^2 \sim n/(\log n)^2 \quad \text{and} \quad \limsup_{n \to \infty} \mathbb{P}(\max_{1 \leq i \leq n} |\tilde{\xi}_i| \geq \varepsilon \sigma_n) = 1.
\]

However, when \(\mathbb{E}(\|\tilde{\xi}_0\|^{2+\delta}) < \infty\) and \(\sigma_n^2 = nh(n)\) the functional CLT holds. Woodroofe-Wu (2004) and also Merlevède-P (2006),
Regular weights and infinite variance (long memory).

\[ a_n = n^{-\alpha} L(n), \quad \text{where } 1/2 < \alpha < 1, \]
\[ \mathbb{E}(\xi_0^2 I(|\xi_0| \leq x)) = H(x) \]

**Example**

Fractionally integrated processes. For \(0 < d < 1/2\) define

\[ X_k = (1 - B)^{-d} \xi_k = \sum_{i \geq 0} a_i \xi_{k-i} \quad \text{where} \quad a_i = \frac{\Gamma(i + d)}{\Gamma(d) \Gamma(i + 1)} \]

and \(B\) is the backward shift operator, \(B\varepsilon_k = \varepsilon_{k-1}\).

For any real \(x\), \(\lim_{n \to \infty} \Gamma(n + x)/n^x \Gamma(n) = 1\) and so

\[ \lim_{n \to \infty} a_n / n^{d-1} = 1 / \Gamma(d). \]
Regularly varying weights and infinite variance; normalizers.

Define \( b = \inf \{ x \geq 1 : H(x) > 0 \} \)

\[
\eta_j = \inf \{ s : s \geq b + 1, \ H(s)/s^2 \leq j^{-1} \}, \quad j = 1, 2, \ldots 
\]

\[
B_n^2 := c_\alpha H_n n^{3-2\alpha} L^2(n) \text{ with } H_n = H(\eta_n)
\]

where

\[
c_\alpha = \left\{ \int_0^\infty [x^{1-\alpha} - \max(x - 1, 0)^{1-\alpha}]^2 dx \right\} / (1 - \alpha)^2.
\]
Invariance principle for regular weights and infinite variance (long memory).

\[ a_n = n^{-\alpha}L(n), \text{ where } 1/2 < \alpha < 1, \ n \geq 1, \ \mathbb{E}(\bar{\xi}_0^2 I(|\bar{\xi}_0| \leq x)) = H(x), \]

\[ L(n) \text{ and } H(x) \text{ are both slowly varying at } \infty. \]

**Theorem**

*(P-Sang 2011)* Define \( W_n(t) = S_{[nt]} / B_n \). Then, \( W_n(t) \) converges weakly to the fractional Brownian motion \( W_H \) with Hurst index \( 3/2 - \alpha \), \( (1/2 < \alpha < 1) \).

Fractional Brownian motion with Hurst index \( 3/2 - 2\alpha \), i.e. is a Gaussian process with covariance structure \( \frac{1}{2} (t^{3-2\alpha} + s^{3-2\alpha} - (t-s)^{3-2\alpha}) \) for \( 0 \leq s < t \leq 1 \).
Selfnormalized invariance principle. Regular varying case

**Theorem**

(P-Sang 2011) Under the same conditions we have

\[
\frac{1}{nH_n} \sum_{i=1}^{n} X_i^2 \xrightarrow{P} A^2 \quad \text{where} \quad A^2 = \sum_i a_i^2
\]

and therefore

\[
\frac{S_{[nt]}}{na_n \sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow \frac{\sqrt{c_\alpha}}{A} W_H(t).
\]

In particular

\[
\frac{S_n}{na_n \sqrt{\sum_{i=1}^{n} X_i^2}} \Rightarrow N(0, \frac{c_\alpha}{A^2}).
\]
Application to unit root testing

\[ Y_n = \rho Y_{n-1} + X_n \quad \text{for} \quad n \geq 1 \]

where \( Y_0 = 0 \) and \( (X_n)_{n\geq1} \) is a stationary sequence and \( \rho \) is a constant. The ordinary least squares (OLS) estimator of \( \rho \) is

\[ \hat{\rho}_n = \frac{\sum_{k=1}^{n} Y_k Y_{k-1}}{\sum_{k=1}^{n} Y_{k-1}^2} . \]

To test \( \rho = 1 \) against \( \rho < 1 \), a key step is to derive the limit distribution of the Dickey–Fuller (DF) test statistic

\[ \hat{\rho}_n - 1 = \frac{\sum_{k=1}^{n} Y_{k-1}(Y_k - Y_{k-1})}{\sum_{k=1}^{n} Y_{k-1}^2} . \]
Application to unit root testing

Theorem

(P-Sang 2012) Assume that $(X_n)_{n \geq 1}$ is a linear process, $I(x) = \mathbb{E}(\xi_0^2 I(\xi_0 \leq x))$ is a slowly varying function at $\infty$.

(a) \[ \frac{\sum_{k=1}^{n} Y_{k-1}^2}{n^3 a_n^2 \sum_{i=1}^{n} X_i^2} \Rightarrow \frac{c_\alpha}{A^2} \int_0^1 W_H^2(t) \, dt \, . \]

(b) \[ \frac{\sum_{k=1}^{n} Y_{k-1} (Y_k - Y_{k-1})}{n^2 a_n^2 \sum_{i=1}^{n} X_i^2} \Rightarrow \frac{c_\alpha W_H^2(1)}{2A^2} . \]

(c) \[ n(\hat{\rho}_n - 1) \Rightarrow \frac{W_H^2(1)/2}{\int_0^1 W_H^2(t) \, dt} . \]
Exact asymptotics

We aim to find a function $N_n(x)$ such that, as $n \to \infty$,

$$\frac{\mathbb{P}(S_n \geq x\sigma_n)}{N_n(x)} = 1 + o(1), \text{ with } \sigma_n^2 = \|S_n\|_2^2.$$

where $x = x_n \geq 1$ (Typically $x_n \to \infty$).

We call $\mathbb{P}(S_n \geq x_n\sigma_n)$ the probability of moderate or large deviation probabilities depending on the speed of $x_n \to \infty$. 
**Exact asymptotics versus logarithmic**

**Exact approximation** is more accurate and holds under less restrictive moment conditions than the logarithmic version

\[
\frac{\log \mathbb{P}(S_n \geq x\sigma_n)}{\log N_n(x)} = 1 + o(1).
\]

For example, suppose \( \mathbb{P}(S_n \geq x\sigma_n) = 10^{-4} \) and \( N_n(x) = 10^{-5} \); then their logarithmic ratio is 0.8, which does not appear to be very different from 1, while the ratio for the exact version is as big as 10.
Theorem

(Nagaev, 1979) Let \((\xi_i)\) be i.i.d. with

\[
P(\xi_0 \geq x) = \frac{h(x)}{x^t} \quad \text{as} \quad x \to \infty \quad \text{for some} \quad t > 2,
\]

and for some \(p > 2\), \(\xi_0\) has absolute moment of order \(p\). Then

\[
P(\sum_{i=1}^{n} \xi_i \geq x\sigma_n) = (1 - \Phi(x))(1 + o(1)) + nP(\xi_0 \geq x\sigma_n)(1 + o(1))
\]

for \(n \to \infty \) and \(x \geq 1\).
Notice that in this case

\[ N_n(x) = (1 - \Phi(x)) + n\mathbb{P}(\xi_0 \geq x\sigma_n). \]

If \( 1 - \Phi(x) = o\left[n\mathbb{P}(\xi_0 \geq x\sigma_n)\right] \) then in we can also choose \( N_n(x) = 1 - \Phi(x) \).

If \( n\mathbb{P}(\xi_0 \geq x\sigma_n) = o(1 - \Phi(x)) \) we have \( N_n(x) = n\mathbb{P}(\xi_0 \geq x\sigma_n) \).

The critical value of \( x \) is about \( x_c = (2 \log n)^{1/2} \).
Let \((\xi_i)\) be i.i.d. with

\[ P(\xi_0 \geq x) = \frac{h(x)}{x^t} \quad \text{as} \quad x \to \infty \quad \text{for some} \quad t > 2, \]

and for some \(p > 2\), \(\xi_0\) has absolute moment of order \(p\).

**Theorem**

*(P-Sang-Zhong-Wu, 2012)* Let \(S_n = \sum_{i=1}^{n} X_i\) where \(X_i\) is a linear process. Then, as \(n \to \infty\),

\[ P \left( S_n \geq x\sigma_n \right) = (1 + o(1)) \sum_{i=-\infty}^{\infty} P( b_{n,i} \xi_0 \geq x\sigma_n ) + (1 - \Phi(x))(1 + o(1)) \]

holds for all \(x > 0\) when \(\sigma_n \to \infty\), \(\sum_{k=-\infty}^{\infty} a_k^2 < \infty\) and \(b_{nj} > 0\),

\[ b_{n,j} = a_{j+1} + \cdots + a_{j+n}. \]
Zones of moderate and large deviations

Define the Lyapunov’s proportion

\[ D_{nt} = B_{n2^{-t/2}} B_{nt} \text{ where } B_{nt} = \sum_i b_{ni}^t. \]

For \( x \geq a(\ln D_{nt}^{-1})^{1/2} \) with \( a > 2^{1/2} \) we have

\[ \mathbb{P}(S_n \geq x\sigma_n) = (1 + o(1)) \sum_{i=1}^{k_n} \mathbb{P}(c_{ni} \xi_0 \geq x\sigma_n) \text{ as } n \to \infty. \]

On the other hand, if \( 0 < x \leq b(\ln D_{nt}^{-1})^{1/2} \) with \( b < 2^{1/2} \), we have

\[ \mathbb{P}(S_n \geq x\sigma_n) = (1 - \Phi(x))(1 + o(1)) \text{ as } n \to \infty. \]
Value at risk (VaR) and expected shortfall (ES) are equivalent to quantiles and tail conditional expectations. Under the assumption $\lim_{x \to \infty} h(x) \to h_0 > 0$

$$\mathbb{P}(S_n \geq x\sigma_n) = (1 + o(1)) \frac{h_0}{x^t} D_{nt} + (1 - \Phi(x))(1 + o(1)).$$

Given $\alpha \in (0, 1)$, let $q_{\alpha,n}$ be defined by $\mathbb{P}(S_n \geq q_{\alpha,n}) = \alpha$. $q_{\alpha,n}$ can be approximated by $x\alpha \sigma_n$ where $x = x_\alpha$ is the solution to the equation

$$\frac{h_0}{x^t} D_{nt} + (1 - \Phi(x)) = \alpha.$$
Extension to dependent structures

- CLT for stationary and ergodic differences innovations with finite second moment. (P-Utev, 2006)


- CLT stationary martingales differences with infinite second moment plus a mild mixing assumption. (P-Sang 2011)

Results for mixing sequences under various mixing assumptions.
Some open problems

Is the CLT for linear processes equivalent with its selfnormalized version?

\[
\frac{S_n}{V_n} \rightarrow N(0, 1) \text{ where } V_n^2 = \sum_{i=1}^{n} X_i^2
\]

CLT for linear processes with infinite variance and ergodic martingale innovations

Functional CLT for linear processes with i.i.d. innovations finite second moment and \( \text{var}(S_n) = nh(n) \)
(necessary and sufficient conditions on the constants)

The same question for generalized martingales

Exact asymptotics for classes of Markov chains

More classes of functions of linear processes


Peligrad, M.; Sang, H.; Zhong, Y.; Wu, W.B.. Exact Moderate and Large Deviations for Linear Processes (2012);
Key Ingredients for exact deviations.

Lemma

Assume $S_n = \sum_{j=1}^{k_n} X_{nj}$ (where $X_{nj}$ represents a triangular array of independent variables) is stochastically bounded, the variables are centered, and $x_n \to \infty$. Then for any $0 < \eta < 1$, and $\varepsilon > 0$ such that $1 - \eta > \varepsilon$,

$$\mathbb{P}(S_n \geq x_n) = \mathbb{P}(S_n(\varepsilon x_n) \geq x_n) + \sum_{j=1}^{k_n} \mathbb{P}(X_{nj} \geq (1 - \eta)x_n)$$

$$+ o\left(\sum_{j=1}^{k_n} \mathbb{P}(X_{nj} \geq \varepsilon x_n)\right) + \sum_{j=1}^{k_n} \mathbb{P}((1 - \eta)x_n \leq X_{nj} < (1 + \eta)x_n).$$
Fuk–Nagaev inequality (S. Nagaev, 1979)

**Theorem**

Let $Y_1, Y_2, \cdots, Y_n$ be independent random variables and $m \geq 2$. Suppose $E[Y_i] = 0$, $i = 1, \cdots, n$, $\beta = m/(m + 2)$, and $\alpha = 1 - \beta = 2/(m + 2)$. For $y > 0$, define $Y^{(y)}_i = Y_i I(Y_i \leq y)$,

$$A_n(m; 0, y) := \sum_{i=1}^{n} E[Y_i^m I(0 < Y_i < y)]$$ and $$B_n^2(-\infty, y) := \sum_{i=1}^{n} E[Y_i^2 I(Y_i < y)].$$

Then for any $x > 0$ and $y > 0$

$$P\left(\sum_{i=1}^{n} Y_i^{(y)} \geq x\right) \leq \exp\left(-\frac{\alpha^2 x^2}{2e^m B^2(-\infty, y)}\right) + \left(\frac{A(m; 0, y)}{\beta xy^{m-1}}\right)^{\beta x/y}.$$
Theorem

Let \((X_{nj})_{1 \leq j \leq k_n}\) be an array of row-wise independent centered random variables. Let \(p > 2\) and denote \(S_n = \sum_{j=1}^{k_n} X_{nj}, \sigma_n^2 = \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^2 \to \infty, M_{np} = \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^p I(X_{nj} \geq 0) < \infty, L_{np} = \sigma_n^{-p} M_{np}\) and denote

\[
\Lambda_n(u, s, \epsilon) = \frac{u}{\sigma_n^2} \sum_{j=1}^{k_n} \mathbb{E}X_{nj}^2 l(X_{nj} \leq -\epsilon \sigma_n / s).
\]

Furthermore, assume \(L_{np} \to 0\) and \(\Lambda_n(x^4, x^5, \epsilon) \to 0\) for any \(\epsilon > 0\). Then if \(x \geq 0\) and \(x^2 - 2 \ln(L_{nt}^{-1}) - (t - 1) \ln \ln(L_{nt}^{-1}) \to -\infty, we have

\[
P(S_n \geq x\sigma_n) = (1 - \Phi(x))(1 + o(1)).
\]