Emergence of collective dynamics from pure noise

(sorry for the change of title!)

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Mathematical Modelling of Complex Systems,
Centrale-Supélec, Châtenay-Malabry, 14-16 dec. 2016.

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ii) some models of inviscid chemotaxis generalizing the inviscid Burgers equation in higher dimensions,
ii) the classical Newtonian model of gravitation (*).

(*) This last statement might be considered as a VERY ROUGH CARICATURE of popular claims, in High Energy Physics, that the Einstein equation is just emerging from quantum fluctuations of vacuum.

AN EXAMPLE OF RANK-BASED DYNAMICS IN 1D

Consider N taxpayers labelled by $\alpha \in \{1, \cdots, N\}$.
$Z_n(\alpha) \geq 0$ is the taxable income of year n.
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Model: $\boxed{Z_{n+1}(\alpha) = Z_n(\alpha) \exp(r\tau) \exp(-\mathcal{G}(\sigma_n)\tau)}$ with a uniform growth rate r for all incomes and a tax rate $\mathcal{G}$ that depends only on the rank.

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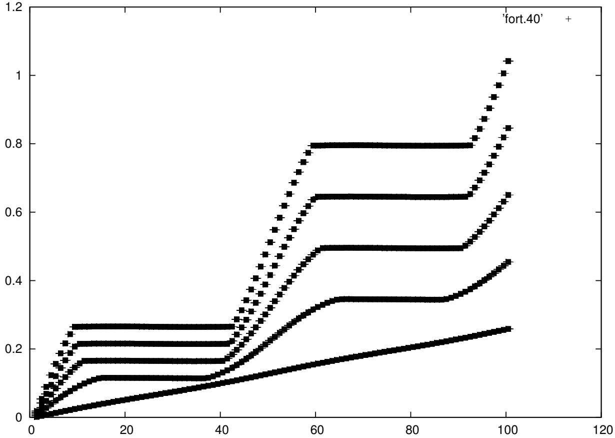
This can be related to hyperbolic scalar conservation laws, the formation of shock waves corresponding to the emergence of classes.

Example: formation of 2 classes

Evolution of the income distribution, starting from a linear profile, with formation of two classes (i.e. two "shocks" in terms of conservation laws).
(Data: $N = 100$, $\tau = 0,01$, $F(u) = u + \frac{sin(4\pi u)}{4}$, $u \in [0,1]$, $t \in [0,1]$, $\tau = 0,01$.)



RANK BASED DYNAMICS IN 1D: (old) RESULTS

For the slightly more general model (with pseudo-noise in option)

$$X_{n+1}(\alpha) = X_n(\alpha) + \tau\ F(w) + (-1)^{(N-1)w}\sqrt{2\eta\tau}\ R(w), \quad w = \frac{\sigma_n(\alpha) - 1}{N - 1}$$

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1) Asymptotic behavior $\tau << 1$, $N >> 1$, **for** $u_n(x) = \frac{1}{N}\sum_{\alpha=1}^{N} 1_{\{x > X_n(\alpha)\}}$

$$\partial_t u + \partial_x(f(u)) = \eta\, \partial_{xx}(r(u)), \quad F(u) = f'(u), \quad R(u) = r'(u) \geq 0$$

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2) A unique "class" emerges whenever $\forall u \in]0,1[,\ f(u) > f(0) = f(1)$.

Y.B. CRAS 1981-82, SINUM 1984, thèse d'état 1986, J. Comp. Appl. Math. 1990.

CORE OF THE TALK: STOCHASTIC ORIGIN OF RANK-BASED DYNAMICS AND NEWTONIAN GRAVITATION

We consider a finite (or periodic) cubic lattice $\{A(\alpha) \in \mathbb{R}^d, \ \alpha = 1, \cdots, N\}$ subject to a small brownian agitation

$$\boxed{Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N}$$

We call "point cloud at time t" the collection of positions $\{Y_t(\alpha)\}$ reached by these particles, disregarding their label $\alpha \in \{1, \cdots, N\}$.

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We call "point cloud at time t" the collection of positions $\{Y_t(\alpha)\}$ reached by these particles, disregarding their label $\alpha \in \{1, \cdots, N\}$. In other words, the cloud lives in $(\mathbb{R}^d)^N / \mathcal{S}_N$, where $\mathcal{S}_N$ is the symmetric group (of all permutations of the N first integers).

WHERE IS THE BROWNIAN CLOUD AT TIME T?

At a fixed time $T > 0$, the probability for the brownian cloud

$$\{A(\alpha) + \sqrt{\epsilon} B_T(\alpha), \quad \alpha = 1, \cdots, N\}$$

to be observed at $X_T = (X_T(\alpha),\ \alpha = 1, \cdots, N) \in \mathbb{R}^{dN}$ has density

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$Z = N!\sqrt{2\pi\epsilon T}^{Nd}$, $\quad |\cdot|$ **and** $||\cdot|| =$ **euclidean norms in** $\mathbb{R}^d$ **and** $\mathbb{R}^{Nd}$**.**

Here, we crucially used that the particles are indistinguishable!!!

VANISHING NOISE AND APPARENT MOTION

$$-\lim_{\epsilon\to 0} \epsilon \, \log \frac{1}{Z} \sum_{\sigma\in\mathcal{S}_N} \exp\left(-\frac{||X_T - A_\sigma||^2}{2\epsilon T}\right) = \frac{1}{2T} \inf_{\sigma\in\mathcal{S}_N} ||X_T - A_\sigma||^2$$

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As a simple consequence of the "large deviation principle", we can show that, as $\epsilon \to 0$, the observer at time T feels that the particles have travelled along straight lines by "optimal transport"

$$X_t = (1 - \frac{t}{T})A_{\sigma_{opt}} + \frac{t}{T} X_T , \quad \sigma_{opt} = \mathrm{Argsup}_{\sigma\in\mathcal{S}_N} \ ((X_T, A_\sigma)), \quad t \in [0, T]$$

LAW AND DISORDER!

From the apparent motion of the cloud up to time T

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This leads to the apparent "law"

$$\frac{dX_t}{dt} = \frac{X_t - A_{\sigma_{opt}}}{t} \ , \quad \sigma_{opt} = \text{Arginf}_{\ \sigma \in \mathcal{S}_N} \ ||X_t - A_\sigma||^2, \quad t \in]0, T]$$

just resulting of the observation of a purely random motion!

ZELDOVICH MODEL AND INVISCID CHEMOTAXIS

$t = e^\theta$ **leads to** $$\boxed{\frac{dX_\theta}{d\theta} = X_\theta - A_{\sigma_{opt}}\ ,\ \ \sigma_{opt} = \mathrm{Arginf}_{\ \sigma \in \mathcal{S}_N}\ \ ||X_\theta - A_\sigma||^2}$$

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Using optimal transport tools, we find, as formal continuous limit,

$$\boxed{\partial_\theta\rho - \nabla\cdot(\rho\nabla_x\varphi) = 0,\ \ \det(I + D^2_x\varphi) = \rho;\ \ \rho \geq 0,\ \varphi\in\mathbb{R},\ (\theta, x)\in\mathbb{R}^{1+d}}$$

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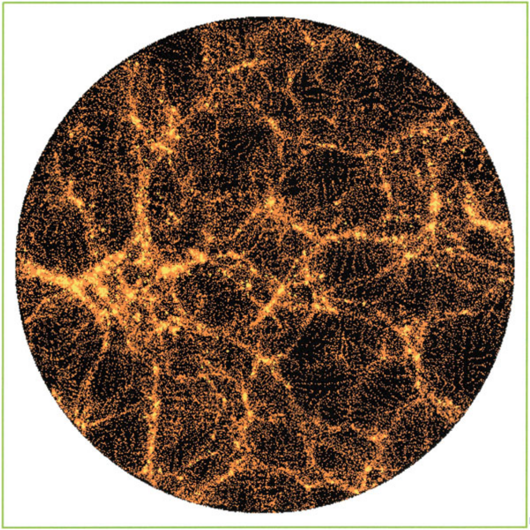
This is a multidimensional generalization of the rank based dynamics discussed at the beginning of this talk. It is equivalent to the Zeldovich model (1970) in Cosmology. It can also be seen as a fully nonlinear version of the (inviscid) chemotaxis model: $\boxed{\partial_\theta\rho - \nabla\cdot(\rho\nabla_x\varphi) = 0,\ \ \triangle\varphi = \rho - \overline{\rho},\ \ \overline{\rho} = \int\rho(t,x)dx = 1}$

Monge-Ampère gravitation: a simulation of the Zeldovich model



Last part: EN ROUTE TO NEWTON'S GRAVITY

We first observe that the probability density we found for the Brownian point cloud to be found at $X \in \mathbb{R}^{Nd}$ at time $t > 0$

$$\frac{1}{N!\sqrt{2\pi\epsilon t}^{Nd}} \sum_{\sigma \in \mathcal{S}_N} \exp\left(-\frac{||X - A_\sigma||^2}{2\epsilon t}\right), \quad X \in \mathbb{R}^{Nd}$$

is just the solution $\rho(t, X)$ of the heat equation in $\mathbb{R}^{Nd}/\mathcal{S}_N$

$$\frac{\partial \rho}{\partial t}(t, X) = \frac{\epsilon}{2} \triangle \rho(t, X), \quad \rho(t = 0, X) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \delta(X - A_\sigma), \quad X \in \mathbb{R}^{Nd}$$

"SURFING THE HEAT WAVE"

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For arbitrarily chosen position $X_{t_0} \in \mathbb{R}^{Nd}$ at $t_0 > 0$, let us "surf" the "heat wave" by solving the ODE

$$\frac{dX_t}{dt} = v(t,X_t), \quad v(t,X) = -\frac{\epsilon}{2}\nabla_X \log \rho(t,X), \quad t \geq t_0$$

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This is an adaptation of de Broglie's "onde pilote" concept. As a matter of fact, a similar calculation also works for the free Schrödinger equation:

$(i\partial_t + \triangle)\psi = 0, \quad \psi(0, X) = \sum_\sigma \exp(-||X - A_\sigma||^2/a^2), \quad v = \nabla \mathcal{I}m \log \psi$

SURFING THE "HEAT WAVE" SYSTEM ... WITH ADDITIONAL NOISE!

Using $t = e^{2\theta}$, the "heat wave" ODE explicitly reads

$$\frac{dX_\theta}{d\theta} = v_\epsilon(\theta, X_\theta) \ , \quad v_\epsilon(\theta, X) = X - \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp\left(\frac{-||X - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-||X - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}$$

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To get Newton's gravitation, our key idea is now to consider large deviations of this ODE subject to additional noise:

$$\frac{dX_\theta}{d\theta} = v_\epsilon(\theta, X_\theta) + \sqrt{\eta}\frac{dB_\theta}{d\theta}$$

THROUGH LARGE DEVIATION AND LEAST ACTION PRINCIPLES

we end up, as $\epsilon, \eta \to 0$, with the following dynamical system

$$\boxed{\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha)) \;, \quad X_\theta(\alpha) \in \mathbb{R}^d, \;\; \alpha = 1, \cdots, N}$$

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involving, at each time t, a discrete optimal transport problem which leads, in the limit $N \to \infty$, to a Monge-Ampère equation.

WRITTING IN KINETIC TERMS (as $N \to \infty$)

Let $f(\theta, x, \xi)$ be the probability of finding a particle at time θ, point $x \in \mathbb{R}^d$ and velocity $\xi \in \mathbb{R}^d$, in the limit $N \to \infty$. We find:

$$\partial_\theta f(\theta, x, \xi) + \nabla_x \cdot (\xi\, f(\theta, x, \xi)) - \nabla_\xi \cdot (\nabla_x \varphi(\theta, x) f(\theta, x, \xi)) = 0$$

$$\det(\mathbb{I} + D_x^2 \varphi(\theta, x)) = \int_{\mathbb{R}^d} f(\theta, x, d\xi), \quad (\theta, x, \xi) \in \mathbb{R}^{1+d+d}$$

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THANKS FOR YOUR ATTENTION!

reference: Y. B., "A double LD principle for MA gravitation", arXiv 2015

LARGE DEVIATIONS OF THE "HEAT WAVE" ODE

We first pass to the limit $\eta \to 0$, while $\epsilon > 0$ is kept fixed. The large deviation theory tells us that the probability to join point X_{θ_0} at $\theta = \theta_0$ and point X_{θ_1} at later time $\theta = \theta_1$ behaves as

$$\exp(-\frac{\mathcal{A}}{\eta}), \quad \eta \to 0, \quad \mathcal{A} = \frac{1}{2} \int_{\theta_0}^{\theta_1} ||\frac{dX_\theta}{d\theta} - v_\epsilon(\theta, X_\theta)||^2 d\theta$$

where we call $\mathcal{A}$ the Freidlin-Vencel action.

$\Gamma-$LIMIT OF THE VENCEL-FREIDLIN ACTION

We now pass to the $\Gamma-$limit $\epsilon \downarrow 0$ (*) in the Vencel-Freidlin action

$$\mathcal{A} = \frac{1}{2}\int_{\theta_0}^{\theta_1} ||\frac{dX_\theta}{d\theta} - v_\epsilon(\theta, X_\theta)||^2 d\theta,$$

$$v_\epsilon(\theta, X) = -\nabla_X \Phi_\epsilon(\theta, X), \quad \Phi_\epsilon(\theta, X) = \epsilon e^{2\theta} \log \sum_{\sigma \in \mathcal{S}_N} \exp(\frac{-||X - A_\sigma||^2}{2\epsilon e^{2\theta}})$$

noticing that

$$\lim_{\epsilon \downarrow 0} \Phi_\epsilon(\theta, X) = -\frac{1}{2} \inf_{\sigma \in \mathcal{S}_N} \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2$$

(*) thanks to L. Ambrosio, private communication.