Illustrating the coalgebraic method: coinduction, circularity and compositionality

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Motivation

1. Coalgebra as a general theory of dynamical systems
2. Coinduction as the main coalgebraic proof principle
3. Algebra: reducing complexity by compositionality
4. Coalgebra: reducing complexity by recognising circularity
5. Enhanced coinduction:
   exploiting both circularity and compositionality

Overview

1. Moessner's Theorem
2. Streams and coinduction
3. Formalising Moessner's Theorem
4. Proving Moessner's Theorem
5. Discussion

1. Moessner's Theorem

Moessner's Theorem (k = 2)

nat	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$Drop_2$	1		3		5		7		9		11	$\cdots$	
Σ	1	4	9	16	25	36	$\cdots$						
	=												
nat^2	1^2	2^2	3^2	4^2	5^2	6^2	$\cdots$						

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Moessner's Theorem (k = 3)

nat	1	2	3	4	5	6	7	8	9	10	11	12	···
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Σ	1	8	27	64	$\cdots$								
	$=$												
nat^3	1^3	2^3	3^3	4^3	$\cdots$								

Moessner's Theorem (k = 4)

nat	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$Drop_4$	1	2	3		5	6	7		9	10	11	$\cdots$
Σ	1	3	6	11	17	24	33	43	54	$\cdots$		
$Drop_3$	1	3		11	17		33	43		67	81	$\cdots$
Σ	1	4	15	32	65	108	175	$\cdots$				
$Drop_2$	1		15		65		175	$\cdots$				
Σ	1	16	81	256	$\cdots$							
	$=$	1^4	2^4	3^4	4^4	$\cdots$						

Moessner's Theorem (k = 5)

nat	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
Drop$_5$	1	2	3	4		6	7	8	9		11	$\cdots$
Σ	1	3	6	10	16	23	31	40	51	$\cdots$		
Drop$_4$	1	3	6		16	23	31		51	$\cdots$		
etc.							$\cdots$					
	$=$	1^5	2^5	3^5	4^5	$\cdots$						

Moessner's Theorem: history

- Conjectured by A. Moessner (1951),
  first proved by O. Perron (1951),
  generalised by I. Paasche (1952) and H. Salie (1952).
- Proof in functional programming by R. Hinze (2008, 2011).
- First coinductive proof by M. Niqui and J.R. (2011).
- New proof using multivariate generating functions,
  by D. Kozen and A. Silva (2013).
- Formalisation in COQ
  of the coinductive proof of M. Niqui and J.R.,
  by R. Krebbers, L. Parlant and A. Silva (2016).

Moessner's Theorem: history

- Today: a new coinductive proof (J.R. 2016, unpublished).
- Very simple, a student's exercise.
- We prove that streams **are** the same by showing that they **behave** the same.
- Cf. classical proofs use complicated bookkeeping, involving binomial coefficients and falling factorials.

2. Our tool: streams and coinduction

Streams of natural numbers

$$\begin{array}{c} \mathbb{N}^\omega \\ \big\downarrow \langle \mathsf{head}, \mathsf{tail} \rangle \\ \mathbb{N} \times \mathbb{N}^\omega \end{array}$$

where

$$\begin{aligned} \mathsf{head}(\sigma) &= \sigma(0) \\ \mathsf{tail}(\sigma) &= (\sigma(1), \sigma(2), \sigma(3), \ldots) \end{aligned}$$

for any stream $\sigma = (\sigma(0), \sigma(1), \sigma(2), \ldots) \in \mathbb{N}^\omega$.

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which we will typically write as

$$\mathsf{head}(\sigma) = \sigma(0) \qquad \textbf{(initial value)}$$

$$\mathsf{tail}(\sigma) = \sigma' \qquad \textbf{(derivative)}$$

Finality of streams

$$
\begin{array}{ccc}
X & \overset{\exists!\, h}{- - - - - - \rightarrow} & \mathbb{N}^\omega \\
{\scriptstyle \forall\, \langle \mathrm{out}, \mathrm{tr} \rangle} \downarrow & & \downarrow {\scriptstyle \langle \mathrm{head}, \mathrm{tail} \rangle} \\
\mathbb{N} \times X & - - - - \rightarrow & \mathbb{N} \times \mathbb{N}^\omega
\end{array}
$$

The function h, defined by

$$h(x) = (\mathrm{out}(x),\ \mathrm{out}(\mathrm{tr}(x)),\ \mathrm{out}(\mathrm{tr}(\mathrm{tr}(x))),\ \ldots)$$

is the *unique* function making the diagram commute.

Streams and bisimulation

A relation $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a **stream bisimulation** if

$$\begin{array}{ccccc}
\mathbb{N}^\omega & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & \mathbb{N}^\omega \\
\downarrow & & \exists\, \downarrow \gamma & & \downarrow \\
\mathbb{N} \times \mathbb{N}^\omega & \longleftarrow & \mathbb{N} \times R & \longrightarrow & \mathbb{N} \times \mathbb{N}^\omega
\end{array}$$

Equivalently, $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation if for all $(\sigma, \tau) \in R$:

$$\begin{array}{lll}
(i) & \sigma(0) = \tau(0) & \text{and} \\
(ii) & (\sigma', \tau') \in R &
\end{array}$$

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Theorem [**Coinduction** proof principle]

Let $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ be a bisimulation. For all streams $\sigma, \tau \in \mathbb{N}^\omega$,

$$(\sigma, \tau) \in R \;\Rightarrow\; \sigma = \tau$$

Proof: straightforward, by showing that $\sigma(n) = \tau(n)$, for all $n \geq 0$, by induction on n. □

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Example

Define

$$\text{zip} : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega \qquad \text{even} : \mathbb{N}^\omega \to \mathbb{N}^\omega \qquad \text{odd} : \mathbb{N}^\omega \to \mathbb{N}^\omega$$

by

$$\begin{aligned} \text{zip}(\sigma, \tau) &= (\sigma(0), \tau(0), \sigma(1), \tau(1), \sigma(2), \tau(2), \ldots) \\ \text{even}(\sigma) &= (\sigma(0), \sigma(2), \sigma(4), \ldots) \\ \text{odd}(\sigma) &= (\sigma(1), \sigma(3), \sigma(5), \ldots) \end{aligned}$$

Their initial values and derivatives satisfy:

$$\begin{aligned} \text{zip}(\sigma, \tau)(0) &= \sigma(0) & \text{zip}(\sigma, \tau)' &= \text{zip}(\tau, \sigma') \\ \text{even}(\sigma)(0) &= \sigma(0) & \text{even}(\sigma)' &= \text{even}(\sigma'') \\ \text{odd}(\sigma)(0) &= \sigma(1) & \text{odd}(\sigma)' &= \text{odd}(\sigma'') \end{aligned}$$

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Example: a proof by coinduction

Proposition: for all $\sigma, \tau \in \mathbb{N}^\omega$, $\text{even}(\text{zip}(\sigma,\tau)) = \sigma$

Proof: we define

$$R = \{ \langle \text{even}(\text{zip}(\sigma,\tau)),\ \sigma \rangle \mid \sigma,\tau \in \mathbb{N}^\omega \}$$

and prove that R is a **bisimulation.** First note that

$$(i) \quad \text{even}(\text{zip}(\sigma,\tau))(0) = \text{zip}(\sigma,\tau)(0) = \sigma(0)$$

Then observe that

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3. Formalising Moessner's Theorem

Moessner's theorem (k = 3)

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On the left, we have:

$$\text{nat} = (1, 2, 3, \ldots)$$

$$\text{nat}^3 = (1^3, 2^3, 3^3, \ldots) = \text{nat} \odot \text{nat} \odot \text{nat}$$

with

$$\sigma \odot \tau = (\sigma(0) \cdot \tau(0),\ \sigma(1) \cdot \tau(1),\ \sigma(2) \cdot \tau(2),\ \ldots)$$

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On the right, we have:

$$\Sigma\,\sigma = (\sigma(0),\ \sigma(0)+\sigma(1),\ \sigma(0)+\sigma(1)+\sigma(2),\ \ldots)$$

$$D_2\,\sigma = (\sigma(0), \sigma(2), \sigma(4), \ldots)$$

$$D_3\,\sigma = (\sigma(0), \sigma(1), \sigma(3), \sigma(4), \sigma(6), \sigma(7), \ldots)$$

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$$D_3\,\sigma = (\sigma(0), \sigma(1), \sigma(3), \sigma(4), \sigma(6), \sigma(7), \ldots)$$

A more convenient formulation

$$\begin{aligned} \text{nat}^3 &= \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\text{nat}) \\ &= \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1}) \end{aligned}$$

where

$$\overline{1} = (1, 1, 1, \ldots)$$

since

$$\Sigma \circ D_4\,(\overline{1}) = \Sigma\,(\overline{1}) = \text{nat}$$

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4. Proving Moessner's Theorem

A proof by coinduction

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\bar{1})$$

The aim is to construct a **bisimulation** relation containing the pair

$$\langle\ \mathrm{nat}^3\,,\ \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\bar{1})\ \rangle$$

Towards that end, let us investigate
the **derivatives** of the streams and operators above.

(**Initial values** will all be straightforward.)

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Inspecting derivatives

For the stream $\mathsf{nat} = (1, 2, 3, \ldots)$, we have

$$\begin{aligned} \mathsf{nat}' &= (2, 3, 4, \ldots) \\ &= (1+1,\ 1+2,\ 1+3,\ \ldots) \\ &= (1, 1, 1, \ldots) \oplus (1, 2, 3, \ldots) \\ &= \overline{1} \oplus \mathsf{nat} \end{aligned}$$

where $\oplus$ denotes the elementwise sum of streams.

Inspecting derivatives

For the product $\sigma \odot \tau$, we have

$$\begin{aligned}(\sigma \odot \tau)' &= (\sigma(0) \cdot \tau(0),\ \sigma(1) \cdot \tau(1),\ \sigma(2) \cdot \tau(2),\ \ldots)' \\ &= (\sigma(1) \cdot \tau(1),\ \sigma(2) \cdot \tau(2),\ \sigma(3) \cdot \tau(3),\ \ldots) \\ &= \sigma' \odot \tau'\end{aligned}$$

Inspecting derivatives

These properties of nat' and $(\sigma \odot \tau)'$ imply:

$$
\begin{aligned}
(\mathsf{nat}^3)' &= (\mathsf{nat} \odot \mathsf{nat} \odot \mathsf{nat})' \\
&= \mathsf{nat}' \odot \mathsf{nat}' \odot \mathsf{nat}' \\
&= (\bar{1} \oplus \mathsf{nat}) \odot (\bar{1} \oplus \mathsf{nat}) \odot (\bar{1} \oplus \mathsf{nat}) \\
&= \binom{3}{0} \cdot \bar{1} \;\oplus\; \binom{3}{1} \cdot \mathsf{nat} \;\oplus\; \binom{3}{2} \cdot \mathsf{nat}^2 \;\oplus\; \binom{3}{3} \cdot \mathsf{nat}^3
\end{aligned}
$$

using some elementary properties of $\oplus$ and $\odot$,
and defining $k \cdot \sigma$ by

$$k \cdot \sigma = (k \cdot \sigma(0),\ k \cdot \sigma(1),\ k \cdot \sigma(2),\ \ldots)$$

Inspecting derivatives

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

So for the stream on the left, we have:

$$(\mathrm{nat}^3)' = \binom{3}{0} \cdot \overline{1} \;\oplus\; \binom{3}{1} \cdot \mathrm{nat} \;\oplus\; \binom{3}{2} \cdot \mathrm{nat}^2 \;\oplus\; \binom{3}{3} \cdot \mathrm{nat}^3$$

Inspecting derivatives

$$\mathrm{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

Turning to the right hand side, we observe:

$$\overline{1}' = \overline{1}$$

Inspecting derivatives

For the drop operators, we have

$$\begin{aligned}(D_2\,\sigma)' &= (\sigma(0), \sigma(2), \sigma(4), \ldots)' \\ &= (\sigma(2), \sigma(4), \sigma(6), \ldots) \\ &= D_2\,\sigma''\end{aligned}$$

And, similarly,

$$(D_3\,\sigma)^{(2)} = D_3\,\sigma^{(3)}$$
$$(D_4\,\sigma)^{(3)} = D_4\,\sigma^{(4)}$$

where the repeated derivatives are defined as usual:

$$\sigma^{(0)} = \sigma$$
$$\sigma^{(k+1)} = (\sigma^{(k)})'$$

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Inspecting derivatives

$$
\begin{aligned}
(\Sigma\,\sigma)' &= (\sigma(0),\ \sigma(0)+\sigma(1),\ \sigma(0)+\sigma(1)+\sigma(2),\ \ldots)' \\
&= (\sigma(0)+\sigma(1),\ \sigma(0)+\sigma(1)+\sigma(2),\ \ldots) \\
&= (\sigma(0),\sigma(0),\sigma(0),\ldots) \quad \oplus \\
&\quad\ (\sigma(1),\ \sigma(1)+\sigma(2),\ \sigma(1)+\sigma(2)+\sigma(3),\ \ldots) \\
&= \overline{\sigma(0)} \oplus \Sigma\,(\sigma')
\end{aligned}
$$

where

$$
\overline{\sigma(0)} = (\sigma(0),\sigma(0),\sigma(0),\ldots)
$$

Inspecting derivatives

Together, these properties imply:

$$
\begin{aligned}
&(\ \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})\ )' \\
&= \binom{3}{0} \cdot\ \overline{1} \\
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(The details would fill 1 or 2 additional slides.)

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Proving Moessner's theorem (k = 3)

$$\text{nat}^3 = \Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})$$

All in all, we have found:

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All in all, we have found:

$(\text{nat}^3)'$	$(\,\Sigma \circ D_2 \circ \Sigma \circ D_3 \circ \Sigma \circ D_4\,(\overline{1})\,)'$	M3′
$= \binom{3}{0} \cdot \overline{1}$	$= \binom{3}{0} \cdot \overline{1}$	M0
$\oplus \binom{3}{1} \cdot \text{nat}$	$\oplus \binom{3}{1} \cdot \Sigma \circ D_2\,(\overline{1})$	M1
$\oplus \binom{3}{2} \cdot \text{nat}^2$	$\oplus \binom{3}{2} \cdot \Sigma \circ D_2 \circ \Sigma \circ D_3\,(\overline{1})$	M2
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Moessner's theorem: the general case

$$\mathrm{nat}^k = \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})$$

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Moessner's theorem: the general case

And so we define $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ by

$$R = \{ \langle \mathrm{nat}^k , \ \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}(\overline{1}) \rangle \mid k \geq 0 \}$$

Is R a **bisimulation relation**?

No, but almost: R is a bisimulation relation **up to sum**!

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Bisimulations up to sum

A relation $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a bisimulation relation **up to sum** if, for all $(\sigma, \tau) \in R$,

(i) if $(\sigma, \tau) \in R$ then $\sigma(0) = \tau(0)$

(ii) there are $n_1, \ldots, n_l \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_l, \tau_1, \ldots, \tau_l \in \mathbb{N}^\omega$ such that

$$\sigma' = n_1 \cdot \sigma_1 \oplus \cdots \oplus n_l \cdot \sigma_l$$
$$\tau' = n_1 \cdot \tau_1 \oplus \cdots \oplus n_l \cdot \tau_l$$

and

$$(\sigma_1, \tau_1) \in R \, , \; \ldots \; , \; (\sigma_l, \tau_l) \in R$$

Coinduction up to sum

Theorem

Let $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ be a bisimulation **up to sum.**

$$\forall \sigma, \tau \in \mathbb{N}^\omega : \quad (\sigma, \tau) \in R \;\Rightarrow\; \sigma = \tau$$

Proof: We define $R^c \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ as the smallest relation s.t.

1. $R \subseteq R^c$
2. if $(\sigma, \tau) \in R^c$ then $(n \cdot \sigma, n \cdot \tau) \in R^c$ (all $n \in \mathbb{N}$)
3. if $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in R^c$ then $(\sigma_1 \oplus \sigma_2, \tau_1 \oplus \tau_2) \in R^c$

It is easy to see that R^c is an (ordinary) bisimulation.
Now the theorem follows by (ordinary) coinduction. □

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is a bisimulation up to sum:

$$\begin{array}{llr}
(\mathrm{nat}^k)' & (\,\Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})\,)' & \mathrm{Mk}' \\
= \binom{k}{0} \cdot \overline{1} & = \binom{k}{0} \cdot \overline{1} & \mathrm{M0} \\
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\oplus \cdots & \oplus \cdots & \cdots \\
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$\oplus \binom{k}{k} \cdot \mathrm{nat}^k$	$\oplus \binom{k}{k} \cdot \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\, (\bar{1})$	Mk

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Derivatives in a picture

$$\sigma \longrightarrow \sigma' \longrightarrow \sigma^{(2)} \longrightarrow \sigma^{(3)} \longrightarrow \cdots$$

More generally, if

$$\sigma' = n_1 \cdot \sigma_1 \oplus \ \cdots \ \oplus n_l \cdot \sigma_l$$

then we will write

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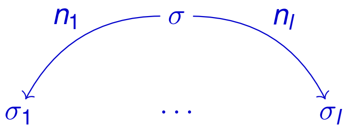

The heart of the matter: circularity

Since

$$\overline{1}' = (1, 1, 1, \ldots)' = \overline{1}$$

we write:

$$\overline{1} \longrightarrow \overline{1} \longrightarrow \overline{1} \longrightarrow \cdots$$

or, equivalently,

$$\overline{1} \circlearrowleft$$

The heart of the matter: circularity

Since for the stream $\mathsf{nat} = (1, 2, 3, \ldots)$, we have

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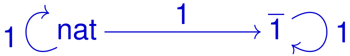

The heart of the matter: circularity

Since

$$(\mathsf{nat}^k)' = \binom{k}{0} \cdot \overline{1} \oplus \binom{k}{1} \cdot \mathsf{nat}^1 \oplus \cdots \oplus \binom{k}{k} \cdot \mathsf{nat}^k$$

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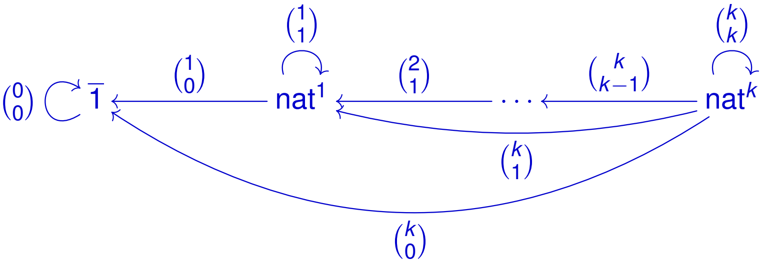
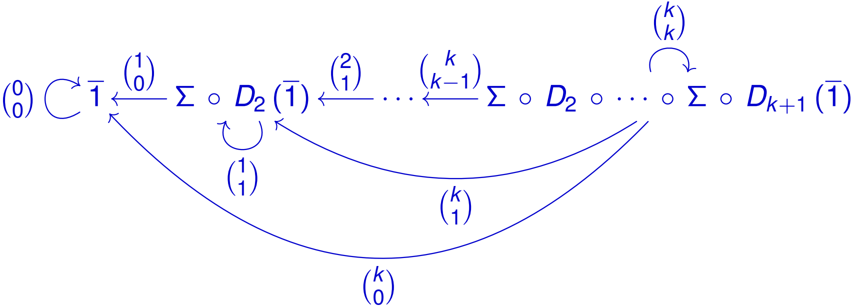
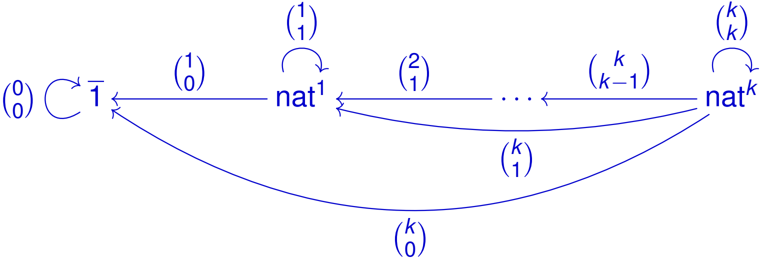


The heart of the matter: circularity

And similarly, we have found




$\binom{0}{0}$
$\overline{1}$
$\binom{1}{0}$
$\binom{1}{1}$
nat^1
$\binom{2}{1}$
$\cdots$
$\binom{k}{k-1}$
$\binom{k}{k}$
nat^k
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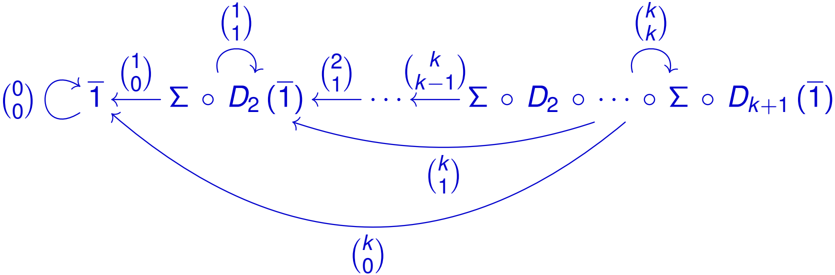
$\binom{k}{0}$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$\Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}(\overline{1})$
$\binom{k}{1}$
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The proof of Moessner, in other words

$$\mathsf{nat}^k \;=\; \Sigma \circ D_2 \circ \cdots \circ \Sigma \circ D_{k+1}\,(\overline{1})$$

Both streams **are** the same ...

because they **behave** the same ...

that is, because they have the same **circular behaviour**:

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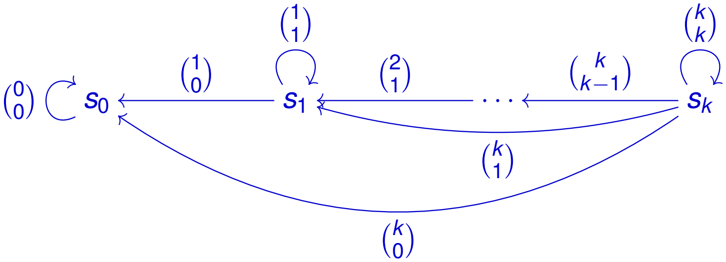

5. Discussion

- We take streams σ as **basic entities**, instead of focussing on their individual **elements** $\sigma(n)$.
- This **prevents** lots of **unnecessary bookkeeping** (cf. binomial coefficients).
- **Complexity** is tackled by recognising **circularity**, with the help of **compositionality**.
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